Impact of Dual Stock Holding and Stochastic Income on the Investor's Remuneration Package

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How to cite this paper: Moalosi-Court, K.I., Lungu, E.M. and Offen, E.R. (2021) Impact of Dual Stock Holding and Stochastic Income on the Investor's Remuneration Package. *Journal of Mathematical Finance*, **11**, 206-217.
https://doi.org/10.4236/jmf.2021.112011

Received: June 2, 2020
Accepted: April 3, 2021
Published: April 6, 2021

Abstract

In this paper, we have investigated how an investor's income, who is rewarded by managing dual company stocks and additionally receives stochastic income, grows. We have calculated the optimal stock price $\bar{S}^*(t) = (S_1^*(t), S_2^*(t))$ and the optimal stock output $\bar{Q}^*(t) = (Q_1^*(t), Q_2^*(t))$ that maximize his returns. We have shown that the best strategy is to choose $S_1^*(t) = S_2^*(t)$ from the set of prices $\bar{S}^*(t) = (S_1^*(t), S_2^*(t))$.

Keywords

Stochastic Income, Investment Value Function, Remuneration

1. Introduction

In this paper, we derive the return for an investor who is rewarded with company stock number of $\alpha_i, i = 1, 2$ units for managing two non-traded geometric Brownian motion risk assets $S_i(t), i = 1, 2$ and also receives stochastic income which accrues at the rate $n\mathcal{L}(t, Y_t) \geq 0$, by trading in options. We want to investigate to what extent a stock based compensation for an investor who also earns stochastic income at time $t \in [0, T]$ that outperforms the strictly stock units compensation scheme.

This study was motivated by the work of Henderson [1] who examined a dynamic portfolio choice for an investor receiving a stream of income rate over a finite investment horizon. The income rate in [1] was stochastic and it was imperfectly correlated to the stock, and the investor could not replicate income risk with the stock and bond assets alone. This led to an investigation that how non-tradibility of the income affects optimal allocation of wealth between the stock and bond. Henderson [1] concluded that this approach allowed more flexibility as most of the parameters could be chosen without resorting to vigorous estimation techniques.

On the other hand, Choon and Manoyios [2], used a similar income rate as Henderson [1], examined the hedging of a stochastic stream in an incomplete market and concluded that when the income rate is modified by $n \in (-1,1)$, then there is reliability in the results.

Other authors who have studied the impact of stochastic income are; Kronborg [3], who considered the optimal consumption and investment problem for an investor endowed with a deterministic stochastic income. The others are Doctor and Offen [4] who incoorporated stochastic income and considered an optimal problem for an investor for different utilities. The study [4] described how an investor can adjust the Merton portfolio through an interpolating hedging demand, in reaction to the stochastic income. Wang, *et al.* [5] studied optimal consumption and savings with stochastic income subject to elastic inter temporal substitution, and concluded that higher risk aversion increases savings and lowers the consumption.

Other works that motivated the approach used in this paper are Chen [6] and Huisman, *et al.* [7]. Both ([6] [7]) applied the method of value matching and smooth pasting conditions to find the optimal investment threshold and the investment value function of an investor. As a sequel to Chen [6] and Huisman, *et al.* [7], we want to find the optimal investment thresholds of $S_i(t)$ for $i = 1,2$, and the expected investment value function for an investor who is compensated with a number of company stock units $\alpha_i, i = 1,2$.

This paper is organized as follows: In Section 2 we formulate the model which is analyzed and simulated in Section 3. Section 4, we calculate the investors remuneration package.

2. The Model

Consider an investor who manages two company assets $S_i(t), i = 1,2$ assumed to evolve as;

$$\frac{\mathrm{d}S_i(t)}{S_i(t-)} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}B_i(t) + \eta_i \mathrm{d}N_i(t), \text{ for } i = 1,2 \tag{1}$$

where $B_i(t)$ is a standard Brownian motions, μ_i, σ_i are constants and η_i is a random jump-amplitudes. We interpret $S_i(t-)$ to mean that whenever there is a jump, the value of the process before the jump is used to ensure that one jump does not make the underlying asset worthless, and $N_i(t)$ is an independent Poisson jump processes with jump rate λ_i, that is, $N_i(t)$ describes the jump times of $S_i(t)$, with

$$\mathbb{P}(\mathrm{d}N_i(t) = n) = \frac{\exp(-\lambda_i \mathrm{d}t)(\lambda_i \mathrm{d}t)^n}{n!}, \ \forall n \in \mathbb{Z}, \ t \in \mathbb{R}^+.$$

and we assume that the processes $N_i(t)$ and $B_i(t)$ are independent, implying that $\mathbb{E}\left[\mathrm{d}N_i(t)\mathrm{d}B_i(t)\right]=0$, and

$$\mathrm{d}N_i(t)=\begin{cases}1 & \text{with probability } \lambda_i\mathrm{d}t\\ 0 & \text{with probability } 1-\lambda_i\mathrm{d}t\end{cases} \tag{2}$$

Suppose the investor is also endowed with a non-negative and non-traded continuous time income rate $\mathcal{L}(t,Y(t))$ at time *t*. The income rate is stochastic and state variable $Y(t)$ evolves as follows:

$$\mathrm{d}Y(t)=\mu(Y(t))\mathrm{d}t+\sigma(Y(t))\mathrm{d}\tilde{B}(t),\ t\geq 0 \tag{3}$$

where $\tilde{B}(t)=\rho B_i(t)+\sqrt{1-\rho^2}B(t)$ and $\rho\in(-1,1)$ is the correlation coefficient with $B_i(t)$, and the process $B(t)$ is a Brownian motion defined on $(\Omega,\mathcal{F},P)$ probability space, $\sigma(Y)$ and $\mu(Y)$ are constants. For $|\rho|<1$, the filtration $\mathcal{F}$ is not generated by $B_i(t)$, implying that the market is incomplete and the claim cannot be perfectly hedged via $S_i(t)$.

Remark 2.1. *Further, we notice that, when* $\sigma(Y)=0$, *then the stochastic income behaves like a riskless asset, and when* $\rho=1$ *then the stochastic income is related to the stock.*

However, the stochastic income defined in Equation (2) not bounded, as such the process $Y(t)$ grows uncontrollably large. Therefore, we assume that it is bounded below at the stochastic income rate $n\mathcal{L}(t,Y(t))\geq 0$, where $n\in\mathbb{Z}$. For $\mathbb{Z}^+$, the investor receives an income stream and for $\mathbb{Z}^-$ he receives nothing. This choice of $n\in\mathbb{Z}$ holds because the process $Y(t)$ is continuous, non negative, allows no arbitrage and allows flexibility in modeling as most of the parameters can be estimated [8].

Remark 2.2.

- It is interesting to observe that the stochastic income can be thought as a future value payment or claim that pays the future value $\int_t^T \mathrm{e}^{r(T-s)}n\mathcal{L}(s,Y_s)\mathrm{d}s$ at terminal time *T*; by taking the sum of the income stream which pays at a rate of $n\mathcal{L}(t,Y_t)$ from time *t* to terminal time *T*.
- We also observe that *n* gives the weight to $\mathcal{L}(t,Y_t)$, but does not change the behavior of the function itself. That is why our study differs from Choon and Manoyios [2] who defined $n\in(-1,1)$ (We refer the reader to Appendix 9 for more information).

In the next section, we find the investors optimal investment threshold and investment value function.

3. The Investors Investment Value Function

Let $S_i(t), i=1,2$, defined in Equation (1) be company assets for which the investor is rewarded $\alpha_i\geq 0$ units in each stock at terminal time $t=T$ for the management of these stocks. He is not allowed to exercise the right to trade these units before $t=T$ avoid inside trading and related arbitrage opportunities (see Kovaleva [9]). Although the investor could under certain circumstances short sell these units, the units remain purely non-traded assets. The reward of assets is then embedded in a remuneration package to be determined later depending on performance.

The objective of this investor is to optimize the company stock value which ultimately increases his wealth through rewards of $\alpha_i \geq 0$ units of shares. Because he cannot trade his assets he is subjected to risks such as market fluctuation and other company risks that affect the magnitude of the reward.

Let

$$P(t) = S_1(t)\alpha_1(a - Q_1(t)) + S_2(t)\alpha_2(a - Q_2(t)) = \sum_{i=1}^{2} \bar{S}(t)\alpha_i(a - \bar{Q}(t)) \tag{4}$$

be the total price of the investors stock at time t in the market, where $\bar{Q}(t) = (Q_1(t), Q_2(t))$ are the total stock output, a is a given constant, $\bar{S}(t) = (S_1(t), S_2(t))$ are the company assets and $a > \bar{Q}$ to ensure that $P(t)$ is positive. We denote the investment value function of the investor for the risk assets $\bar{S}(t)$ by $\tilde{V}(\bar{S}(t)) = \tilde{V}(\bar{S})$.

Using the standard real option method, the Bellman equation for the value function can be expressed as

$$\varpi\tilde{V}(\bar{S}) = \max\left\{\frac{1}{\mathrm{d}t}\mathbb{E}\left[\tilde{V}(\bar{S})\right]\right\}, \tag{5}$$

where we assume $\varpi > \mu_i$ to be a continuous time discount rate that ensures that stock is exercised within a finite period of time. Using Itos Lemma on Equation (5), we obtain

$$\begin{aligned}\varpi\tilde{V}(\bar{S}) = \sum_{i=1}^{2} \mu_i S_i \frac{\partial\tilde{V}(S_i)}{\partial S_i} + \frac{1}{2}(\sigma_i S_i)^2 \frac{\partial^2\tilde{V}(S_i)}{\partial S_i^2} \\ + \lambda_i\left[V(t,(1+\eta_i)S_i) - V(t,S_i)\right].\end{aligned} \tag{6}$$

Note that $\varpi\tilde{V}(0) = \lambda_i\left[V(t) - V(t)\right] = 0$ implying that if $S_i(t)$ ever goes to zero, the value functions remain zero. This removes arbitrage opportunities.

We look for a solution of Equation (6) of the type

$$V(\bar{S}(t)) = \sum_{i=1}^{2} \bar{A}\bar{S}^{\beta}(t), \quad \beta > 0, \tag{7}$$

where $\bar{A} = (A_1, A_2)$ are positive constants. Substituting Equation (7) into (6) yields the following characteristic equation for β:

$$\begin{aligned}\varpi\sum_{i=1}^{2} A_i S_i^{\beta} = \sum_{i=1}^{2} A_i\mu_i\beta S_i^{\beta} + \sum_{i=1}^{2}\frac{1}{2}\sigma_i^2\beta(\beta-1)A_i S_i^{\beta} \\ + \sum_{i=1}^{2}\lambda_i\left[A_i(1+\eta_i)^{\beta} S_i^{\beta} - A_i S_i^{\beta}\right],\end{aligned} \tag{8}$$

Equation (8) simplifies to

$$\frac{1}{2}\sigma_i^2\beta^2 + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)\beta + \lambda_i(1+\eta_i)^{\beta} - \lambda_i - \varpi = 0. \tag{9}$$

and has solution:

$$\beta = \frac{\frac{1}{2}\sigma_i^2 - \mu_i + \sqrt{\left\{\mu_i - \frac{1}{2}\sigma_i^2\right\}^2 + \sigma_i^2\left(\lambda_i + \varpi - \lambda_i(1+\eta_i)^{\beta}\right)}}{\sigma_i^2} > 1.$$

McDonald and Siegel [10] have argued that the condition $\sigma_i < \mu_i$ ensures that $\beta > 1$ so that the solution is well defined, as such we ignore the other solution of β for $\sigma_i > \mu_i$ since it gives a negative solution.

We solve for the value function in Equation (6) subject to the following boundary conditions:

$$\tilde{V}(0) = 0 \tag{10}$$

$$\tilde{V}(\bar{S}(t)) = \sum_{i=1}^{2} \frac{Q_i(t) S_i(t) \alpha_i (a - Q_i(t))}{\varpi - \mu_i - \lambda_i \eta_i} - \kappa Q_i(t) \tag{11}$$

$$\frac{\partial \tilde{V}(\bar{S}(t))}{\partial \bar{S}(t)} = \sum_{i=1}^{2} \frac{Q_i(t) \alpha_i (a - Q_i(t))}{\varpi - \mu_i - \lambda_i \eta_i}. \tag{12}$$

Condition (10) simply indicates that the value will be 0 if $\bar{S}(0) = 0$, while condition (11) and (12) are the value matching and the smooth pasting conditions to ensure that $\bar{S}(t)$ is optimal and $\tilde{V}(\bar{S}(t))$ can be maximized when the investment is at the threshold $\bar{S}$. The parameter κ in Equation (11) is a unit cost and $\kappa Q_i(t)$ is the total cost with respect to the stock output.

Note that for a positive value function $\kappa Q_i(t) < \frac{Q_i(t) S_i(t) \alpha_i (a - Q_i(t))}{\varpi - \mu_i - \lambda_i \eta_i}$ and $\varpi - \mu_i - \lambda_i \eta_i > 0$ we obtain $\tilde{V}(\bar{S}(t)) > 0$. In simple terms the value matching condition can be seen as the net pay of this investor, where $\frac{Q_i(t) S_i(t) \alpha_i (a - Q_i(t))}{\varpi - \mu_i - \lambda_i \eta_i} > 0$ gives gross value.

We can write the value that the investor can invest only once as

$$\tilde{V}(\bar{S}(t)) = \begin{cases} \sum_{i=1}^{2} A_i S_i^{\beta}(t) & \text{if } \bar{S} < \bar{S}^* \\ \sum_{i=1}^{2} \frac{Q_i(t) S_i(t) \alpha_i (a - Q_i(t))}{\varpi - \mu_i - \lambda_i \eta_i} - \kappa Q_i(t) & \text{if } \bar{S} \geq \bar{S}^* \end{cases} \tag{13}$$

where the investment optimal thresholds $\bar{S}^*(t)$ are to be determined. According to Henderson [8] the solution $\bar{S} < \bar{S}^*$ implies that the investor can invest immediately and $\bar{S} \geq \bar{S}^*$ implies that investing now is subject to risks due to unforeseen market conditions such as volatility, etc.

From Equation (13) we get the investment threshold as:

$$\bar{A} = \sum_{i=1}^{2} \frac{S_i^{1-\beta}(t)}{\beta} \frac{Q_i(t) \alpha_i (a - Q_i(t))}{\varpi - \mu_i - \lambda_i \eta_i} \tag{14}$$

$$\bar{S}(t) = \frac{\beta}{\beta - 1} \sum_{i=1}^{2} \frac{\kappa (\varpi - \mu_i - \lambda_i \eta_i)}{\alpha_i (a - Q_i(t))}, \tag{15}$$

$$\bar{Q}(t) = \frac{1}{\beta - 1} \sum_{i=1}^{2} \frac{(\beta - 1) S_i(t) - \beta \kappa (\varpi - \mu_i - \lambda_i \zeta_i)}{\alpha_i S_i(t)}. \tag{16}$$

Now, to obtain the optimal stock output $\bar{Q}^*(t)$ (we use (14), (15) into (7)), and the optimal investment threshold $\bar{S}^*(t)$, we utilize the value matching and smooth pasting condition of the investor with $\alpha_i \geq 0$ to obtain

$$\begin{cases} \bar{Q}^*(t) = \dfrac{1}{\beta+1}\sum_{i=1}^{2}\alpha_i\left(a - Q_i(t)\right) \\ \bar{S}^*(t) = \dfrac{\beta+1}{\beta-1}\sum_{i=1}^{2}\dfrac{\kappa\left(\varpi-\mu_i-\lambda_i\eta_i\right)}{\alpha_i\left(a-Q_i(t)\right)} \end{cases} \tag{17}$$

The expected optimal investment value function of the investor given the optimal levels of $\bar{S}^*(t)$ is given by:

$$\begin{aligned} \tilde{V}\left(\bar{S}^*(t)\right) &= \left(\frac{\beta-1}{\beta+1}\right)^{\beta}\sum_{i=1}^{2}\left(\frac{\alpha_i\left(a-Q_i(t)\right)S_i(t)}{\kappa\left(\varpi-\mu_i-\lambda_i\eta_i\right)}\right)^{\beta}\frac{\kappa\alpha_i\left(a-Q_i(t)\right)}{\beta^2-1} \\ &= \left(\frac{\beta-1}{\beta+1}\right)^{\beta}\left[\left(\frac{\alpha_1\left(a-Q_1(t)\right)S_1(t)}{\kappa\left(\varpi-\mu_1-\lambda_1\eta_1\right)}\right)^{\beta}\frac{\kappa\alpha_1\left(a-Q_1(t)\right)}{\beta^2-1}\right. \\ &\quad \left.+\left(\frac{\alpha_2\left(a-Q_2(t)\right)S_2(t)}{\kappa\left(\varpi-\mu_2-\lambda_2\eta_2\right)}\right)^{\beta}\frac{\kappa\alpha_2\left(a-Q_2(t)\right)}{\beta^2-1}\right] \end{aligned} \tag{18}$$

Clearly, when $\alpha_i = 0$, for $i = 1,2$, $\tilde{V}\left(\bar{S}^*(t)\right) = 0$, this avoids any arbitrage opportunities. Nevertheless an alternative and generalized methodology can be used to obtain (18) as shown in the Appendix (1).

Proposition 3.1. *The investment value function* $\tilde{V}\left(\bar{S}^*(t)\right)$ *increases as* $\alpha_i \geq 0$.

Proof. We sketch the proof briefly. Since

$$a > Q_i,\ \varpi - \mu_i - \lambda_i\eta_i > 0,$$

then substituting $\bar{S}^*(t) = \left(S_1^*(t), S_2^*(t)\right)$ into Equation (18) we obtain

$$\tilde{V}\left(\bar{S}^*(t)\right) = \sum_{i=1}^{2}\frac{\kappa\alpha_i\left(a-Q_i\right)}{\beta^2-1}, \tag{19}$$

which is linear in both α_1 and α_2. Hence, for either or both $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ the function $\tilde{V}\left(\bar{S}^*(t)\right)$ increases $\alpha_i = 0, i = 1,2$, increases.

We can conclude from proposition (3.1) that the value of the investor increases with the number of shares received.

It is worth noting that, unlike the Black-Scholes formula, the investor has two criteria to meet: first he targets an output $a > Q_i, i = 1,2$ and secondly, he works towards a higher prices for the stocks, $\bar{S} = S_1, S_2$, which in turn increases his stock share $\alpha_i\bar{S}$.

Simulation of the Investment Value Function

So far, we have looked at $\tilde{V}\left(\bar{S}^*(t)\right)$ in general terms. Now in **Figure 1** we want to simulate $\tilde{V}\left(\bar{S}^*(t)\right)$ by varying the values of $\alpha_i, i = 1,2$, to demonstrate Proposition 3.1. The parameter ϖ, λ_i, μ_i and $\eta_i, i = 1,2$ are speculated and have no market value but are used to demonstrate Proposition 3.1. For these simulations, we have used; $\varpi = 1.11$, $\lambda_1 = \lambda_2 = 0.64$, $\mu_1 = \mu_2 = 0.02$ and $\eta_1 = \eta_2 = 0.5$.

Figures 1(a)-(d) illustrate how the value function increases for various values

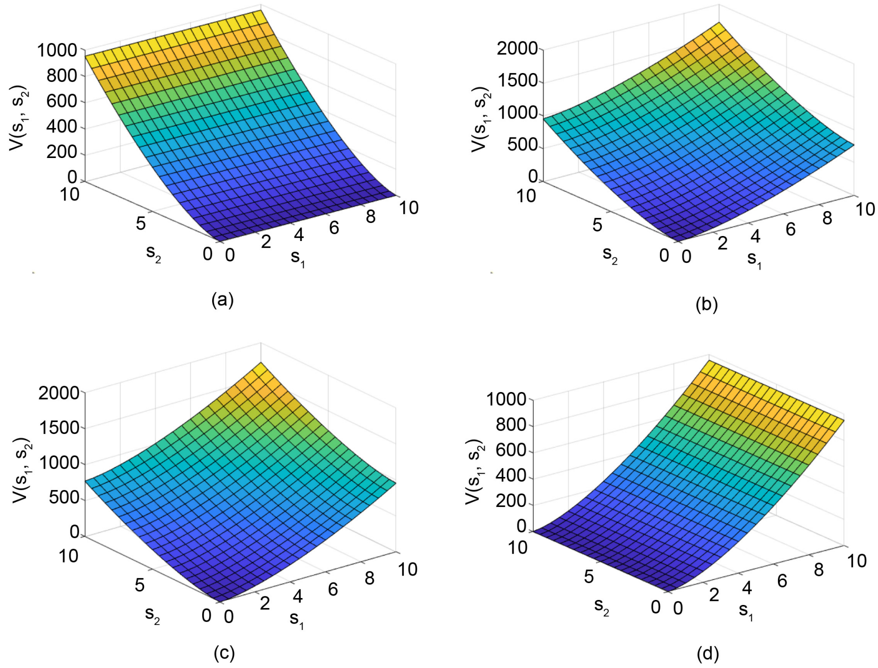

Figure 1. Illustration of how the total investment value function $\tilde{V}\left(\bar{S}^*(t)\right)$ increases for various values of $\alpha_i, i=1,2$. (a) For $S_1(t)$: $\alpha_1=0$ and for $S_2(t)$: $\alpha_2=7$; (b) For $S_1(t)$: $\alpha_1=4$ and for $S_2(t)$: $\alpha_2=7$; (c) For $S_1(t)$: $\alpha_1=7$ and for $S_2(t)$: $\alpha_2=4$; (d) For $S_1(t)$: $\alpha_1=7$ and for $S_2(t)$: $\alpha_2=0$.

of $\alpha_i, i=1,2$. From **Figure 1(a)** and **Figure 1(b)**, we see that an increase in the investors allocation of $S_1(t)$ stock units from $\alpha_1=0$ to $\alpha_1=4$ increases the total investment value. While from **Figure 1(c)** and **Figure 1(d)**, we see that a decrease in the allocation of $S_2(t)$ stock units from $\alpha_2=4$ to $\alpha_2=0$, still brings an increase in the total investment value. Nevertheless, the graphs generally suggest that, as long as units from both stocks are nonzero (*i.e.* $\alpha_1>0$ and $\alpha_2>0$), the investment growth will outperform the reward where the investor is rewarded with units from one stock only.

Figure 2 shows that the higher $\alpha_i, i=1,2$, the higher the return for the investor. However, the investor must ensure that $a \gg Q_i(t), i=1,2$.

It must be pointed out that having these stock units does not necessarily guarantee that the investor will get good returns as company performance can be affected by a number of factors like the market and company risks that are not considered in this study [9].

4. The Investor's Remuneration Package

In this section, we have evaluated the investors remuneration package, and further more investigate the impact of stochastic income on the remuneration package.

A remuneration package can be seen as the investors benefits or rewards when assets are converted in monetary value at time $t=T$. The generalized

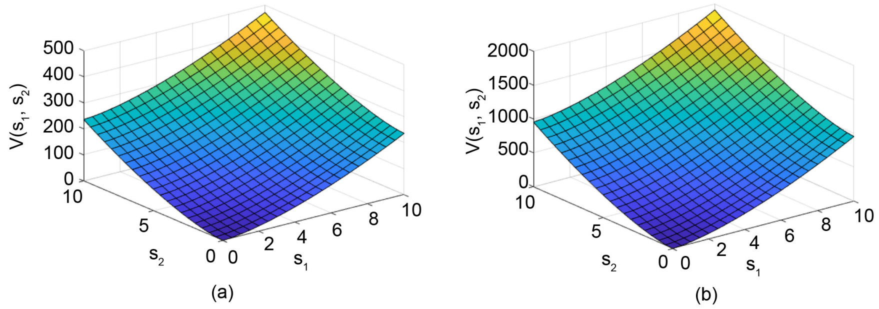

Figure 2. Observation of total investment value function $\tilde{V}(\bar{S}^*(t))$ when $\alpha_1 = \alpha_2$. (a) For $S_1(t)$: $\alpha_1 = 4$ and for $S_2(t)$: $\alpha_2 = 4$; (b) For $S_1(t)$: $\alpha_1 = 7$ and for $S_2(t)$: $\alpha_2 = 7$.

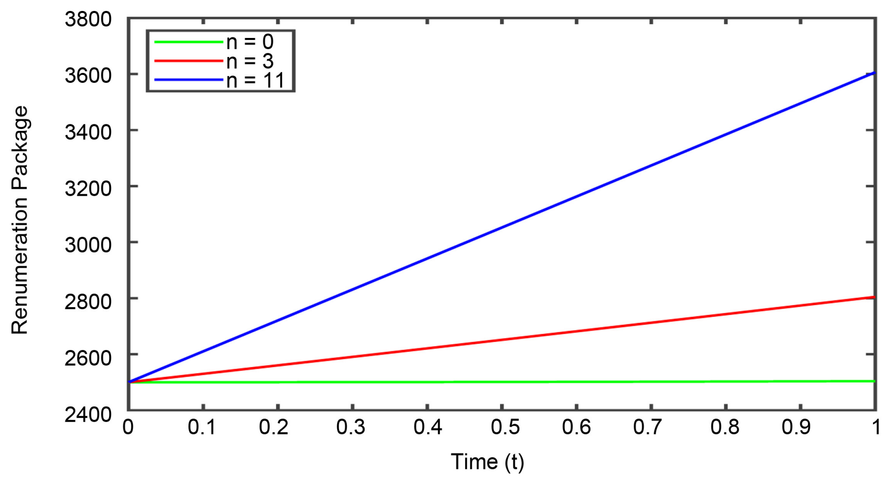

Figure 3. Impact of stochastic income rate on the remuneration package.

expected value function that constitutes the remuneration package of the investor is given by

$$\begin{aligned} & V\left(t, w, \bar{S}(t), Y(t); \alpha_i\right) \\ & = \sup_{(\pi_u)_{u\ge t}} \mathbb{E}\left[U\left(w + \tilde{V}\left(\bar{S}^*(T)\right) + n\mathcal{L}(t, Y_t)\right)\right] \\ & = \sup_{(\pi_u)_{u\ge t}} \mathbb{E}\left[U\left(w + \left(\frac{\beta-1}{\beta+1}\right)^{\beta} \sum_{i=1}^{2}\left(\frac{\alpha_i(a-Q_i)}{\kappa(\varpi-\mu_i-\lambda_i\eta_i)}\right)^{\beta}\right.\right. \\ & \left.\left.\times \frac{\kappa\alpha_i(a-Q_i)(S_i(T))^{\beta}}{\beta^2-1} + \int_0^t e^{r(T-s)} n\mathcal{L}(s, Y_s)\,\mathrm{d}s\right)\right], \end{aligned} \tag{20}$$

where $w > 0$ is the salary, and monetary returns from the stock units and the inflow value from the stochastic income.

Then **Figure 3** gives a general overview of the impact of stochastic income on the remuneration package (given by Equation (20)). Some of the parameter values we used are; $\varpi = 1.11$, $\lambda_1 = \lambda_2 = 0.64$, $\mu_1 = \mu_2 = 0.02$, $\eta_1 = \eta_2 = 0.5$, $\alpha_1 = 4$, $\alpha_2 = 7$, $\mu_y = 0.184$, $\sigma_y = 0.46$ and $\rho = 0.8$. These values were chosen arbitrarily and they give an illustration that support our conclusions.

From **Figure 3**, we observe that the presence of stochastic income increases the investors wealth. Note that the larger the rate of receiving the stochastic income, the bigger net remuneration for the investor. That is, for $n=3$ and $n=11$ we see the presence of stochastic income, while for $n=0$ indicates the absence of stochastic income. For more reading on what the significance of n refer to remark (2.2) and Appendix (6.2).

5. Conclusion

In this paper we derived and analysed the investment value function and remuneration package of an investor who is rewarded company stock units α_i and also receives stochastic income. From our results, we have concluded that the investor has two objectives; first to satisfy the company requirement that $a > Q_i(t)$ and secondly to increase the number of shares α_i given to him as rewards for ensuring that $a \gg Q_i(t)$. The stochastic income has the effect of increasing the investor's wealth but it is the rate at which he receives the stochastic income which matters. At this point, our results and all our analyses can hold in the n-dimension, and we leave that open to the reader for further study.

Acknowledgements

I gratefully acknowledge the funding received from the Simmons Foundation (US) through the Research and Graduate Studies in Mathematics (RGSMA) in Botswana International University of Science and Technology. Many thanks go to my father in heaven who made sure that this paper is completed.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

Alternative Method of Section 3

This alternative, is a generalized version that one can work with. Here we define our variable in terms in matrix form. Hence we redefine our HJB equation as

$$\begin{aligned}\varpi\tilde{V} &= \mu_1 S_1(t)\frac{\partial\tilde{V}}{\partial S_1(t)} + \mu_2 S_2(t)\frac{\partial\tilde{V}}{\partial S_2(t)} \\ &+\frac{1}{2}\sigma_1^2\left(S_1(t)\right)^2\frac{\partial^2\tilde{V}}{\partial\left(S_1(t)\right)^2}+\frac{1}{2}\sigma_2^2\left(S_2(t)\right)^2\frac{\partial^2\tilde{V}}{\partial\left(S_2(t)\right)^2} \\ &+\lambda_1\left[V\left(t,(1+\eta_1)S_1(t)\right)-V\left(t,S_1(t)\right)\right] \\ &+\lambda_2\left[V\left(t,(1+\eta_2)S_2(t)\right)-V\left(t,S_2(t)\right)\right].\end{aligned} \tag{21}$$

Then our boundary conditions are as follows

$$V(S)=0 \tag{22}$$

$$V(S)=\frac{QS\alpha(aI-Q)}{\varpi-\mu_i-\lambda_i\zeta_i}-\Delta Q \tag{23}$$

$$\frac{\partial V(S)}{\partial Q}=\frac{Q\alpha(aI-Q)}{\varpi-\mu_i-\lambda_i\zeta_i} \tag{24}$$

where $S=\begin{pmatrix}S_1 & S^2\end{pmatrix}$, $Q=\begin{pmatrix}Q_2\\ Q_1\end{pmatrix}$, $\Delta Q=\begin{pmatrix}\Delta Q_1 & \Delta Q_2\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Then value is equal to

$$F(S)=AS^{\beta} \tag{25}$$

where $A=\begin{pmatrix}A_1\\ A_2\end{pmatrix}$. Therefore from (23), (25) and equating the $\frac{\partial V(S)}{\partial Q}$ of both equations we obtain threshold S as

$$A=\frac{(S)1-\beta}{\beta}\frac{Q\alpha(aI-Q)}{\varpi-\mu_i-\lambda_i\zeta_i} \tag{26}$$

$$S=\frac{\beta}{\beta-1}\sum_{i=1}^{2}\frac{\varpi-\mu_i-\lambda_i\zeta_i}{\alpha(aI-Q)}, \tag{27}$$

which will yield into an optimal threshold $\tilde{S}$ and optimal stock output Q^*:

$$\begin{cases} Q^*=\dfrac{1}{\beta-1}(aI-Q) \\ \tilde{S}=\dfrac{\beta+1}{\beta-1}(\varpi-\mu_i-\lambda_i\zeta_i) \end{cases}$$

Then the investment value function for $\alpha_i \geq 0$ is

$$\tilde{V}(\tilde{S})=\left(\frac{\beta-1}{\beta+1}\right)^{\beta}\left(\frac{\alpha(aI-Q)}{\varpi-\mu_i-\lambda_i\zeta_i}\right)^{\beta}\frac{\alpha(a-Q)(S)^{\beta}}{(\beta-1)(\beta+1)}. \tag{28}$$

Impact of *n* on the Income Rate

We looked at an investor with income rate $n\mathcal{L}(t,Y_t)\geq 0$, and we assumed that

$n \in \mathbb{Z}$, as indicate in Section 2. **Figure 4** shows that indeed when n is a positive integer the income rate produces positive results, while for $n = 0$ no income is received and when n is negative gives negative income, as such violate the condition of income rate (of being bounded below at zero). This generally shows that n play as the intensity of $\mathcal{L}(t, Y_t)$ and it is not restricted only on $n \in (-1,1)$ as indicated by the works of [2].

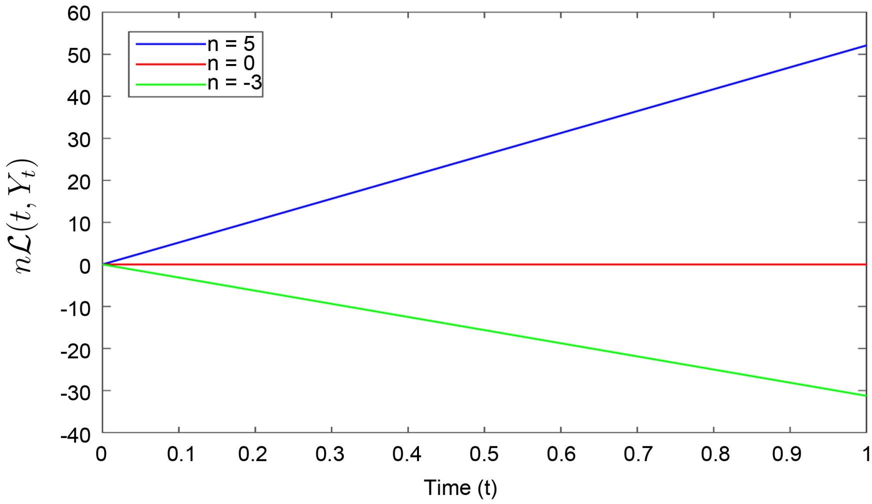

Figure 4. Impact of n on the income rate.