Stochastic Analysis on Optimal Portfolio Selection for DC Pension Plan with Stochastic Interest and Inflation Rate

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Abstract

In this paper, we study optimal investment, consumption and portfolio choice in a framework where the pension planner member (PPM) embarks on an investment policy to cover up for some certain life targets. The aim of the pension plan manager is to maximize the expectation of total wealth at the time of retirement. The investment return process comprises of risk free asset and two risky assets, and the PPM benefit lies in a complete market that is constrained by the inflation rate. Explicit solutions for constant absolute risk aversion utility functions are obtained and optimal strategies are derived by applying by dynamic programming on the Hamilton-Jacobi-Bellman (HJB) equations. Our numerical results show various effects of some economic parameters on the optimal strategies. The inflation price market risk governs the amount invested in both stock and bond, at the same time varying the premium ratio (η), causes effects on the investment returns. We also investigated the effects of the correlation coefficient (ρ) when set high on consumption rate and income rate. Finally a sensitivity analysis is graphically presented.

Keywords

Investment, Consumption, Hamilton-Jacobi-Bellman Equation, Defined Contribution Pension Fund, CARA Utility Function

1. Introduction

The research on pension fund has been discussed around 1900s, the likes of [1], have discussed a problem of Defined Contribution (DC) pension fund in the presence of minimum guarantee. This is where the fund manager invests the initial wealth and the stochastic contribution flow into the financial market.

The following researchers, [2] [3] focused on the optimal portfolio strategies with minimum guarantee and protection in a DC scheme. Bayracktar and Young [4] study the life insurance choice problem from the perspective of two wage earners under exponential utility with a resulting optimal life insurance purchase strategy that is independent of household wealth. Bruhn [5], approaches the question using a power function and show that the optimal death benefit is related to the wealth of the household. In this context, Huang [6] assume a stochastic wage process and a constant relative risk aversion preference. Numerical approaches have been employed in discrete-time models (e.g. [7] [8]) and in continuous-time models (e.g., [6] [9]). In the latter, some special methodologies are used to simplify numerical computing procedures. When the wage process is not stochastic, research on the problem of optimal life insurance, consumption, or investment rules under uncertain lifetime begins with the pioneering work of Yaari [10] and Richard [11]. Pliska and Ye [12] extend Richard's model and relax the assumption that the wage earner's lifetime is bounded by the planning horizon. Duarte [13] continues this strand and extends the model to a multidimensional case with more economic interpretations. Related research on the asset allocation problem includes [14] [15] [16] and [17] to mention a few.

Among other related literature, Fortune [18] applies the expected utility hypothesis of choice under uncertainty to the problem of optimal life insurance, Doherty and Eeckhoudt [19] study optimal insurance without expected utility, and Meier [20] investigates why young people do not buy long-term care insurance. In this article, we assume that preferences exhibit constant absolute risk aversion (as in [4] associated with a mean-reverting stochastic wage process). Constant absolute risk aversion (CARA) preferences are especially useful as benchmark modeling devices for analytical tractability. The study by [1] [21], consists of risk which was only due to stock price market, and [22] in their work included the risk associated with the inflation. This work will follow the same approach as in [22] where their study focused on the proportions to be invested in the stock price and inflation-linked bond, while in this work we put into consideration consumption and income as our additional optimal strategies with more assets.

2. The Model

2.1. The Financial Model

We consider a continuous trading economy over the time period $[0,T]$ characterized by the 2-dimensional Brownian motions $\left(W^S(t),W^I(t)\right)$, defined on a given filtered probability space $\left(\Omega,\mathcal{F},\mathcal{F}_t^S,\mathcal{F}_t^I,P\right)$ where P is the real-world probability measure. $\left\{\mathcal{F}_t^S,\mathcal{F}_t^I\right\}$ are right continuous filtration whose information are generated by two standard Brownian motions $\left(W^S(t),W^I(t)\right)$ whose sources of uncertainties are respectively to the stock market and the inflation rate. The market is assumed to be well defined within interval time, $t\in[0,T]$, where $T>0$ is the terminal time period. The correlation between the two Brownian motions is given by $\mathrm{d}W_I \mathrm{d}W_S = \frac{1}{2}\rho_{SI}\mathrm{d}t$. Let τ be the stopping time, which represents the uncertain life time of the wage earner. τ is a non-negative random variable with probability density function $f(t)$ and distribution function $F(t)$. The survival function is given by $\bar{F}(t) = 1 - F(t)$ and $\lambda(t)$ denotes the hazard rate. Furthermore let $f(s,t)$ denote the conditional probability density for the wage earner to die before time *s*, given that he/she was alive at time *t* and $\bar{F}(s,t) = \exp\left(-\int_t^s u\mathrm{d}u\right)$, indicating the corresponding conditional survival probability. We denote consumption by $c(t)$ and $y(t)$ as the income. It implies the insurance company will pay $\frac{y(t)}{\eta(t)}$ at the point of PPM's death with $\eta(t)$ the premium ratio.

The amount invested in the stock, the bond and the account are respectively, $u_S(t), u_B(t)$ and $-u_S(t) - u_B(t)$ at time *t*.

- The state variables are chosen as assets or the pension plan at any time *t*, that is, $X(t), t \in [0,T]$).
- The decision variables are;

$$\{u_S(t), u_B(t), c(t), y(t)\}, t \in [0,T],$$

are the proportions to be invested in the stock price, bond, as well as considering consumption and income.

Proposition 1. [22] The inflation rate is given by the equation:

$$\mathrm{d}I(t) = \pi_e(t) I(t)\mathrm{d}t + \sigma_I I(t)\mathrm{d}W_I, \quad I(0) = i \tag{2.1}$$

Whose solution is given by

$$I(t) = i\mathrm{e}^{\int_0^t \left(\pi_e(s)\mathrm{d}s - (1/2)\sigma_I^2 t + \sigma_I \mathrm{d}W_I(s)\right)} \tag{2.2}$$

for $\pi_e(t) = r_N(t) - r_R(t)$ as the expected inflation rate which is the difference between the nominal (r_N) and real (r_R) interest rate. The volatility of the inflation is represented by σ_I.

Throughout this paper we consider the Cox-Ingersoll-Ross model, [23] real interest rate given by

$$\mathrm{d}r_R(t) = \left(\alpha - \beta r_R(t)\right)\mathrm{d}t + \sigma_{r_R}\sqrt{r_R}\,\mathrm{d}W_{r_R}, \quad t \geq 0, \tag{2.3}$$

where W_t is the Wiener process (modelling the random market risk factor) and the parameter σ_{r_R} represents the instantaneous volatility of the real interest rate. The parameter α denotes the long term mean level, while β represents the rate of mean reversion. The factor, $\sigma_{r_R}\sqrt{r_R}$, prevents the possibility of negative interest rates for all positive values of α and β. The non-risky asset is denoted by $\left(S_0(t)\right)$ to be determined by

$$\mathrm{d}S_0(t) = r_R(t) S_0 \mathrm{d}t, \tag{2.4}$$

with initial condition $S_0(0) = 1$, where $r_R(t): [0,T] \to R^+$.

Proposition 2. [22] The stock price subject to inflation evolves according to the *Ito*ˆ process:

$$\mathrm{d}S(t) = \left(r_R(t) + \lambda_1\sigma_S^S + \lambda_2\sigma_S^I\theta_I\right)S(t)\mathrm{d}t + \sigma_S^S S(t)\mathrm{d}W_S + \sigma_S^I S(t)\mathrm{d}W_I \tag{2.5}$$

that has the solution:

$$S(t)=\exp\left\{\int_0^t r_R(s)\mathrm{d}s+\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I-\frac{1}{2}\left[\left(\sigma_S^S\right)^2+\left(\sigma_S^I\right)^2+\rho_{SI}\sigma_S^S\sigma_S^I\right]\right)t\right.$$
$$\left.+\sigma_S^S\mathrm{d}W_S(t)+\sigma_S^I S(t)\mathrm{d}W_I(t)\right\}, \tag{2.6}$$

with $S(0)=1$ and θ_I denotes the inflation price market risk. The constants λ_1 and λ_2 are the instantaneous risk premiums associated respectively with the positive volatility, σ_S^S and σ_S^I. *see* **Appendix A** *for proof.*

Proposition 3. [22] The inflation linked bond is described by the stochastic differential equation (SDE):

$$\mathrm{d}B(t,I(t))=(r_R(t)+\sigma_I\theta_I)B(t,I(t))\mathrm{d}t+\sigma_I B(t,I(t))\mathrm{d}W_I(t) \tag{2.7}$$

which has the solution

$$B(t,I(t))=B(0,I(0))\exp\left\{\int_0^t r_R(s)\mathrm{d}s+\left(\sigma_I\theta_I-\frac{1}{2}(\theta_I)^2\right)t+\sigma_I^I\mathrm{d}W_I(t)\right\} \tag{2.8}$$

the proof can be found in **Appendix B**.

2.2. Salary

The wage of the pension fund contributor or client is described by the SDE:

$$\mathrm{d}P(t)=\mu_P P(t)\mathrm{d}t+\sigma_P^S P(t)\mathrm{d}W_S(t)+\sigma_P^I P(t)\mathrm{d}W_I(t) \tag{2.9}$$

where is the expected instantaneous rate of salary. The two volatility scale factors of the stock and inflation are denoted by σ_P^S and σ_P^I respectively.

2.3. Contribution Process

The client has to contribute a certain proportion of his/her salary, and it evolves according to the equation;

$$y(t)=\delta P(t)+\xi(t),\ \text{for } t\in[0,T] \tag{2.10}$$

where $\delta P(t)$ is the proportion of the salary that the client or employee has agreed upon with the employer to be paid towards the pension. The function $\xi(t)$ is a supplementary contribution paid to amortize past and present experience deviations. The supplementary contribution is a deterministic function given by

$$\mathrm{d}\xi(t)=-I\xi(t)\ \text{for } t\in[0,T], \tag{2.11}$$

where I the known inflation rate at that particular time. It is assumed that a large sum amount of money, $X(0)=X_0$ is initially deposited into the pension at time $t=0$. The contribution is a non-negative, progressive process such that,

$$\int_0^T y(t)\mathrm{d}t<\infty,\ \mathbb{P}-a.s.,\ \forall t\in[0,T], \tag{2.12}$$

2.4. Wealth

Let $X(t)$ denote the wealth of the fund at any time, ($t\in[0,T]$ Taking Equations (2.4), (2.5) and (2.7), the wealth process is described by the SDE:

$$\begin{aligned}\mathrm{d}X(t)=&\left(\left[r_R(s)+\begin{pmatrix}u_S(t) & u_B(t)\end{pmatrix}\begin{pmatrix}\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\\ \sigma_I\theta_I\end{pmatrix}\right]X(t)-c(t)-y(t)\right)\\&+\begin{pmatrix}u_S(t) & u_B(t)\end{pmatrix}\begin{pmatrix}\sigma_S^S & \sigma_S^I\\ 0 & \sigma_I\theta_I\end{pmatrix}\begin{pmatrix}\mathrm{d}W_S(t)\\ \mathrm{d}W_I(t)\end{pmatrix}X(t)+X(t)u_B(t)\sigma_I\mathrm{d}W_I(t).\end{aligned}\tag{2.13}$$

The decision variables are $(u_S(t),u_B(t),c(t),y(t))$.

The PPM's legacy will be equal to

$$Z(\tau)=X(\tau)+\frac{y(\tau)}{\eta(\tau)}.$$

2.5. Derivation of the HJB-Equation through Stochastic Dynamic Programming

The utility function is defined as $U(c,t)=\left(\frac{\mathrm{e}^{-\rho t}}{\gamma}\right)c^\gamma, B(Z,t)=\left(\frac{\mathrm{e}^{-\rho t}}{\gamma}\right)Z^\gamma$ and $L(X)=\left(\frac{\mathrm{e}^{-\rho t}}{\gamma}\right)X^\gamma$ for the consumption, the legacy and the terminal wealth respectively. We derive the optimal investment, consumption and allocation of wealth to the risky asset from the following;

$$V(x,t)=\sup_{(u_S(t),u_B(t),c(t),y(t))\in\mathcal{A}}E\left[\int_0^{T\wedge\tau}U(c(s))\mathrm{d}s+B(Z(\tau))_{\{\tau\le T\}}+L(X(T))_{\{\tau\ge T\}}\right]$$

The parameters that the family bearer has to choose are $u_S(t),u_B(t),c(t)$ and $y(t)$. The goal remains the same, which is to optimize the expectation of the utility.

In the case of a financial market with only a risk-free asset, the optimization problem could be solved through dynamic programming by applying the stochastic dynamic programming on $X(t)$ since it follows a stochastic differential equation. Next, the dynamic programming principle is applied which enable us to derive the equation.

Define

$$\begin{aligned}&J(t,x,c,y,u_S,u_B)\\&=E\left[\int_0^{T\wedge\tau}U(c(s))\mathrm{d}s+B(Z(\tau))_{\{\tau\le T\}}+L(X(T))_{\{\tau\ge T\}}\,\vdots\,\tau>t,\mathcal{F}_t^S,\mathcal{F}_t^I\right],\end{aligned}$$

such that $V(t,x)=\sup_{(u_S(t),u_B(t),c(t),y(t))\in\mathcal{A}}J(t,x,c,y,u_S,u_B)$. This optimization problem is again a problem with random terminal time. However, it is expressed as a problem with fixed terminal time,

$$\begin{aligned}&J(t,x,c,y,u_S,u_B)\\&=E\left[\int_0^{T\wedge\tau}\left(f(u,t)B(Z(u))+\bar{F}(u,t)U(c(u))\right)\mathrm{d}u+\bar{F}(T,t)L(X(T))\,\vdots\,\mathcal{F}_t^S,\mathcal{F}_t^I\right],\end{aligned}$$

The stochastic dynamic programming principle for this problem can be stated as follows:

$$V(x,t)=\sup_{(u_S(t),u_B(t),c(t),y(t))\in\mathcal{A}} E\Big[\exp\Big\{-\int_t^s \lambda(v)\,\mathrm{d}v\Big\}V(X(s),s) + \int_t^s \big(f(u,t)B(Z(u))+\bar{F}(u,t)U(c(u))\big)\mathrm{d}u : \mathcal{F}_t^S,\mathcal{F}_t^I\Big], \tag{2.14}$$

for all $0\le t<s<T$. From this stochastic dynamic programming principle we can derive the dynamic programming equation. Consider the wealth process $X(t)$, of which the evolution is described by the differential Equation (2.13).

Applying the Itô's lemma, we get

$$\begin{aligned} &V(T,X(T)) \\ &=V(x,t)+\int_t^T\Big\{V_t(s,X(s))+V_x(s,X(s))\Big[\big\{r_R(s) \\ &+u_S\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)+u_B\sigma_I\theta_I\big\}X(s)-c(s)-y(s)\Big] \\ &+\frac{1}{2}V_{xx}(s,X(s))\Big(\left(u_S\sigma_S^S\right)^2 \\ &+\left(u_S\sigma_S^S\left(u_S\sigma_S^I+u_B\sigma_I\theta_I+u_B\sigma^I\right)\right)\rho_{SI}+\left(u_S\sigma_S^I+u_B\sigma_I\theta_I\right)^2 \\ &+2u_B\sigma^I\left(u_S\sigma_S^I+u_B\sigma_I\theta_I+u_B\sigma^I\right)+\left(u_B\sigma^I\right)^2\Big)\Big\}X^2(s)\mathrm{d}s \\ &+\int_t^T V_x(s,X(s))\left(u_S\sigma_S^I+u_B\sigma_I\theta_I+u_B\sigma^I\right)X(s)\mathrm{d}W_I \\ &+\int_t^T V_x(s,X(s))u_S\sigma_S^S X(s)\mathrm{d}W_S. \end{aligned} \tag{2.15}$$

Let $m_1=\left(\sigma_S^S\right)^2+\left(\sigma_S^I\right)^2+\sigma_S^S\sigma^I\rho_{SI}$,
$m_2=\left(\sigma_S^S\sigma_S^I+\sigma_S^S\sigma_I\theta_I\right)\rho_{SI}+2\left(\sigma_S^I\sigma_I\theta_I+\sigma_S^I\sigma_I\right)$ and
$m_3=\sigma_I\theta_I+2\left(\sigma^I\right)^2\theta_I+\left(\sigma^I\right)^2$

Suppose $V(x,t)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation, then

$$\begin{aligned} 0=&\sup_{(u_S(t),u_B(t),c(t),y(t))\in\mathcal{A}} V_t(x,t)-\lambda V(x,t) \\ &+V_x(s,X(s))\Big(\big\{r_R(t)+u_S\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)+u_B\sigma_I\theta_I\big\}X(t)-c(t)-y(t)\Big) \\ &+\frac{1}{2}V_{xx}(s,X(s))\left(m_1(u_S)^2+m_2u_Su_B+m_3u_B^2\right)X^2+\lambda B(Z(t))-U(c,t) \end{aligned} \tag{2.16}$$

We differentiate Equation (2.16) with respect to the optimal strategies to get

$$0=V_x(x,t)X\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)+\frac{1}{2}V_{xx}(x,t)\left(2m_1u_S+m_2u_B\right)X^2$$

$$0=V_x(x,t)X\left(\sigma_I\theta_I\right)+\frac{1}{2}V_{xx}(x,t)\left(m_2u_S+2m_3u_B\right)X^2$$

$$0=-V_x(x,t)+\mathrm{e}^{-\rho t}c^{\gamma-1}$$

$$0=-V_x(x,t)+\frac{\lambda(t)\mathrm{e}^{-\rho t}}{\mu(t)}\left(x+\frac{y(t)}{\mu(t)}\right)^{\gamma-1}$$

Theorem 4. The optimal amounts spent on stock, bond, consumption and income are given by

$$u_B^* = \frac{x+b(t)}{X(\gamma-1)} C_0 \tag{2.17}$$

$$u_S^* = \frac{x+b(t)}{X(\gamma-1)} C_1 \tag{2.18}$$

$$c^*(t) = a(t)^{\frac{1}{\gamma-1}} (x+b(t)) \tag{2.19}$$

$$y^*(t) = \eta(t) \left\{ \left(\frac{\lambda(t)}{\eta(t)} \right)^{\frac{1}{1-\gamma}} a(t)^{\frac{1}{\gamma-1}} (x+b(t)) - x \right\}, \tag{2.20}$$

where

$$C_0 = \frac{1}{4m_1m_3 - m_2^2} \left(2(\lambda_1\sigma_S + \lambda_2\sigma_I\theta_I) m_2 + 4\sigma_I\theta_I m_1 \right)$$

$$C_1 = -\left(\lambda_1\sigma_S^S + \lambda_2\sigma_S^I\theta_I \right) \left(1 + \frac{m_2^2}{m_1(4m_1m_3 - m_2^2)} \right) + \frac{2\sigma_I\theta_I m_1}{m_1(4m_1m_3 - m_2^2)}$$

$$a(t) = \frac{-\gamma^2}{\gamma-1} \int_t^T \left[\exp\left(\frac{-\gamma}{\gamma-1} \int_t^s \left(-\lambda(v) - \rho + \frac{-\gamma}{\gamma-1} G_1 \right) \mathrm{d}v \right) \exp\left(\frac{\rho}{\gamma-1} s \right) G_2 \right] \mathrm{d}s$$

$$b(t) = \int_t^T -\left(y(s) + x(r_R + \mu) \right) \mathrm{d}s$$

see **Appendix C** *for a detailed proof.*

Example. Numerical example. Our model is constructed for a person's entire lifetime. We used some economic parameters in [22], to evaluate the optimal strategies. It is worth noting that in this manuscript our additional contribution was to introduce consumption and income unlike in [22], they did not focus on consumption.

Table 1 is sampled for the following graphs which show the impact of the economic factor rho on consumption and income, for **Figures 3-6** respectively.

3. Discussion and Analysis

Here is our observation and analysis of the impact for some parameters on our Optimals.

- We have graphs of consumption and income for mu = 0.1 in **Figure 1** (comparison of **Figure 1** and **Figure 2**) and 0.9 in **Figure 2**. The time in years is varied to show how two individuals can differ depending on how they have planned their investments. We see that the investor who prolongs his or her time horizon will have high consumption at old age, and this sometimes happen to most people who choose not to consume much as a way to save for the dependent. We also observe the full consumption for a person who reduced his time horizon. The investor has high consumption around 80 s, this is where the person has high use of money which could be due to high claim of money around late 60 s. Usually this kind individuals are those with lot of dependents and other use of money for example, buying new big houses

Table 1. The Impact of the economic factor on consumption and income.

	Age 20	Age 40	Age 60	Age 80
Results for rho = 0.08				
Consumption	629,300	1,366,000	2,628,000	3,737,000
income	62,940	168,400	380,800	618,000
Results for rho = 0.09				
Consumption	734,500	1,822,000	4,005,000	6,508,000
income	73,590	224,900	580,900	1,077,000
Results for rho = 0.18				
Consumption	29,970	246,900	1,802,000	9,718,000
income	3025	30,580	26,180	1,611,000
Results for rho = 0.2				
Consumption	41,120	442,300	4,214,000	29,680,000
income	4154	54,800	612,400	4,919,000

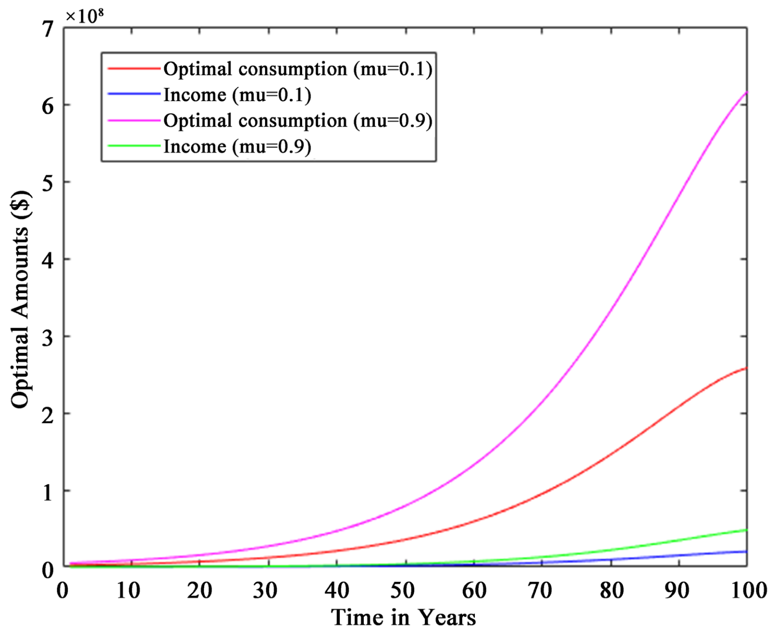

Figure 1. Optimal rates caused by mu = 0.1 and 0.9.

at well-developed cities, traveling to expensive places and buying expensive cars.

- The impact of the correlation coefficient (ρ) on consumption (c) and income (y). The first subgraph shows a wide difference between consumption and income in early years, this is caused by the low correlation coefficient (ρ = 0.08) in **Figure 3**. The same observation occurs on graph with (ρ = 0.09) in

Figure 4 with a slight difference, we see a gap closure from year 1 to year 45. We also observe correlation coefficient (ρ = 0.18) in Figure 5, this clearly shows that the parameter governs the optimal amounts specifically from young age to late 50 s. We finally discovered that as the correlation coefficient varies in an ascending order the consumption rate and income increases as shown in Figure 6 with (ρ = 0.18 vs ρ = 0.2)). It is clear that the use of money will begin at a later age beyond late 30 s. The higher the coefficient the higher the rate of consumption and income.

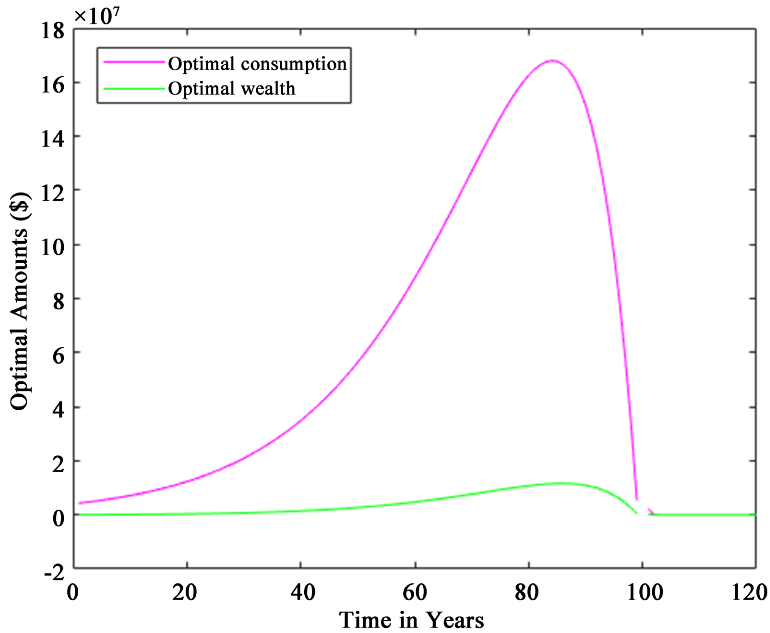

Figure 2. Change of income for mu =0.9 for a whole life.

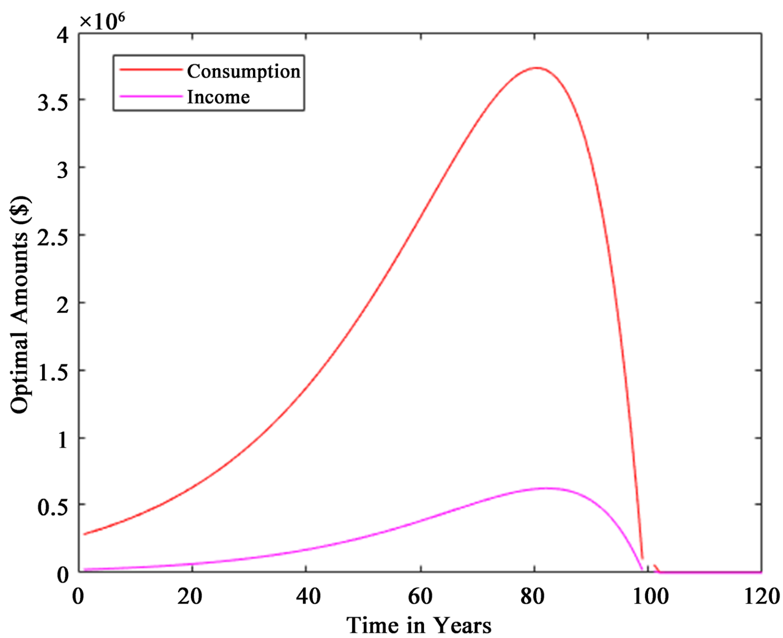

Figure 3. Impact of ρ = 0.08 on consumption and income.

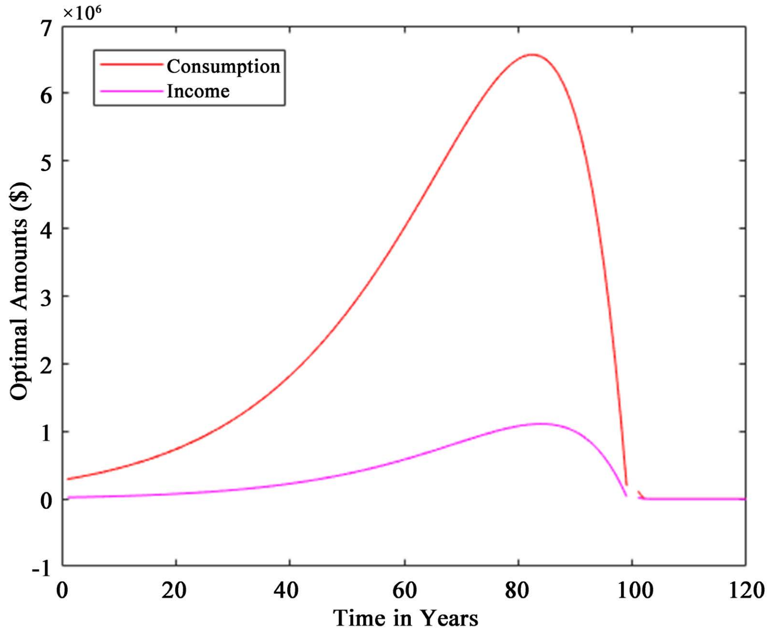

Figure 4. Impact of $\rho = 0.09$ on consumption and income.

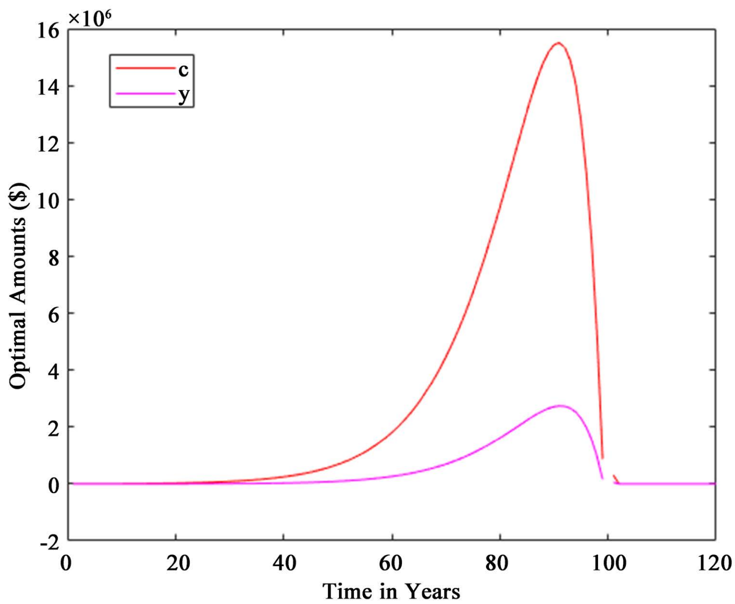

Figure 5. Impact of $\rho = 0.18$ on consumption and income.

- The impact of the economic parameter premium ratio eta $\eta(t)$. In **Figure 7**, we see that the income (y) is affected by the economic parameter premium ratio eta $\eta(t)$ while the rate of consumption (c) is not affected by this parameter. An increase of premium ratio courses a large increase on income, as shown by the graph we have ascending values of the income increasing as eta is increased and the consumption value remains the same for varying eta. In this article we did not restrict our parameter so to avoid unrealistic results it would be good to restrict this economic parameters to some reasonable interval that they suit the daily upgrading financial market.

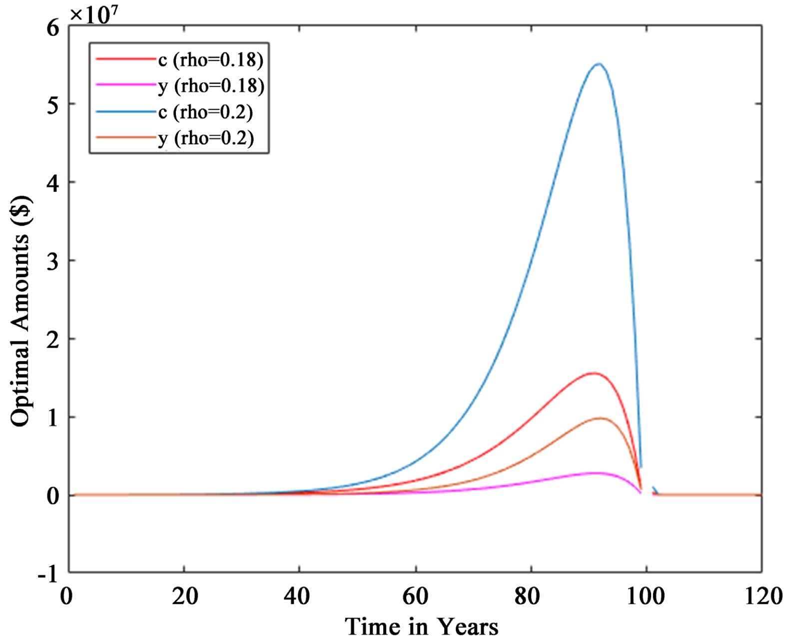

Figure 6. Consumption (c) and income rate (y) of rho=0.18 compared to $\rho = 0.2$ with same initial amount.

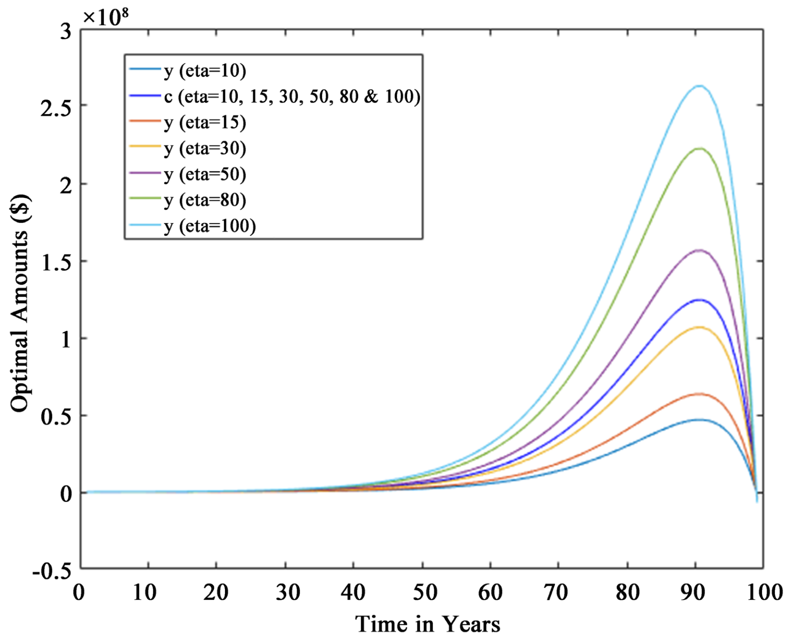

Figure 7. Optimal returns for varrying the economic factor eta ($\eta(t)$) with same initial amount.

- We have graphs of optimal amounts invested in stock $u_S(t)$ and bond $u_B(t)$, **Figures 8-10**. **Figure 8** shows the optimals for the low inflation market risk with high. The models shows that more money should be invested in the bond for low inflation market risk. As the inflation market risk increases we see a turn around, where now the investor will have to put more money in the stock market. The last graph shows that as the inflation price market risk

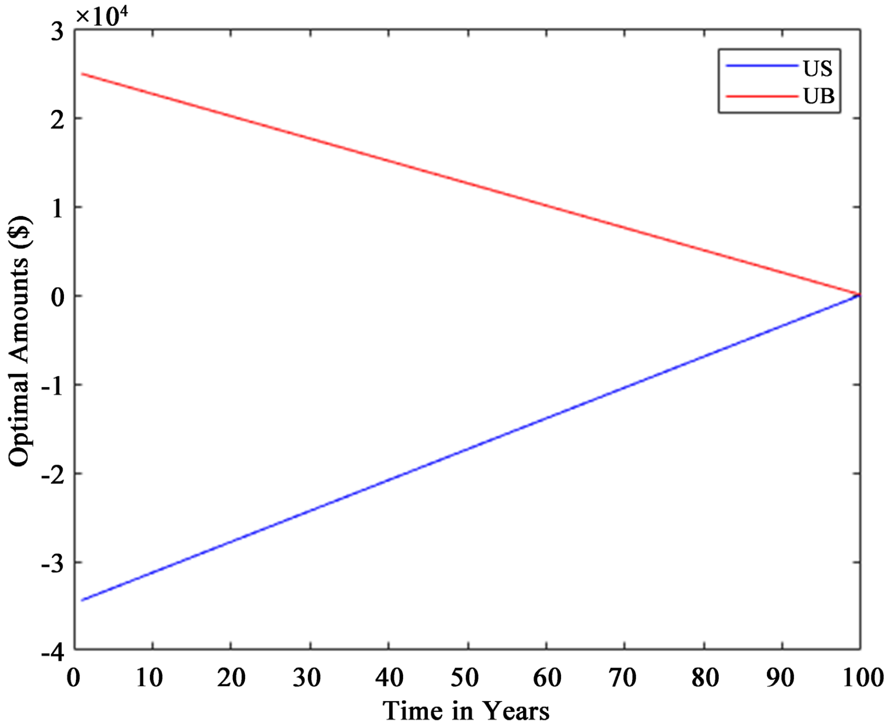

Figure 8. Optimal amount for Stock and Bond with low inflation price market risk and relatively low initial amount.

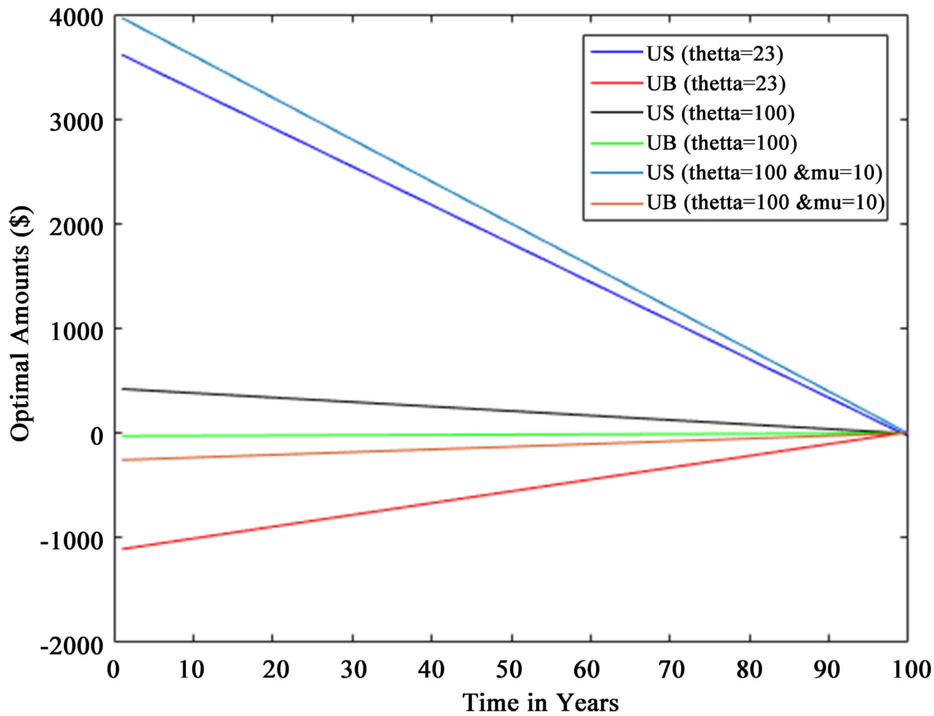

Figure 9. Optimal amount for Stock and Bond with varrying mild inflation price market risk and high initial amount.

θ_I goes higher and higher the optimal amounts in the bond will become zero, while there will be a huge rise on the stock price.

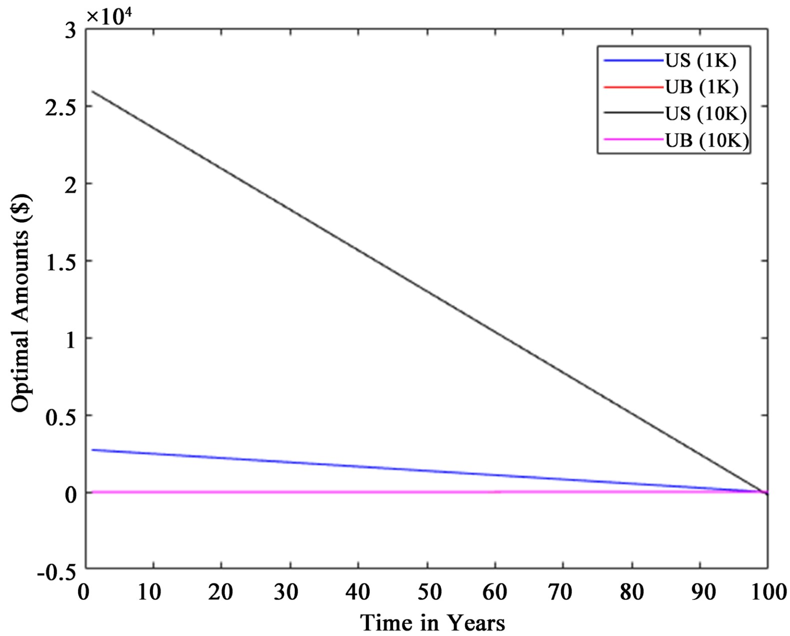

Figure 10. Optimal amount for Stock and Bond with high inflation price market risk and high initial amount.

4. Conclusion

We see that the rate of return in both consumption and income follow a special distribution in statistics and probability namely the beta distribution. This is so because the function of consumption and income has some similar construction as the beta function, which has a component that causes a similar behavior of the graph. These findings are new and this kind of comparison has not been discussed in previous articles, therefore we present these findings as new results and our financial model fits the modern-day system of investment because it is also practical for an individual to invest money only to enjoy more of it at old age usually after retirement. The paper can be extended in future by using the jump diffusion model and the regime change model. The model can also be improved by considering life insurance for an individual or group.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

A. Proof of Proposition 2

Proof. Let

$$f\left(t,S\left(t\right)\right)=\log S\left(t\right)$$

Using Ito's lemma, we have that

$$\begin{aligned}\mathrm{d}\left(\log S\left(t\right)\right)=&\left[\left(r_R\left(t\right)+\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)\mathrm{d}t+\sigma_S^S\mathrm{d}W_S+\sigma_S^I\mathrm{d}W_I\right]\\&+\frac{1}{2}\left[\frac{-1}{S\left(t\right)^2}\right]\left[\mathrm{d}S\left(t\right)\right]^2\end{aligned}\tag{A.1}$$

It can be shown that

$$\left[\mathrm{d}S\left(t\right)\right]^2=\left[\left(\left(\sigma_S^S\right)^2+\left(\sigma_S^I\right)^2\right)+2\sigma_S^S\mathrm{d}W_S\sigma_S^I\mathrm{d}W_I\right]\left[S\left(t\right)\right]^2\tag{A.2}$$

We know that the correlation between the two Brownian motions is given as

$$\mathrm{d}W_I\mathrm{d}W_S=\frac{1}{2}\rho_{SI}\mathrm{d}t\tag{A.3}$$

Therefore, substituting equation (A.3) into equation (A.2) and replacing the results into (A.1), we get:

$$\begin{aligned}\mathrm{d}\left(\log S\left(t\right)\right)=&\left[\left(r_R\left(t\right)+\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)\mathrm{d}t+\sigma_S^S\mathrm{d}W_S+\sigma_S^I\mathrm{d}W_I\right]\\&-\frac{1}{2}\left[\left(\left(\sigma_S^S\right)^2+\left(\sigma_S^I\right)^2\right)+\sigma_S^S\sigma_S^I\rho_{SI}\right]\mathrm{d}t\end{aligned}\tag{A.4}$$

Integrating both sides of the above equation over the interval $\left[0,t\right]$, will lead to the following solution:

$$\begin{aligned}S\left(t\right)=\exp\Bigg[&\int_0^t\left[\left(r_R\left(t\right)+\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)\mathrm{d}t+\sigma_S^S\mathrm{d}W_S+\sigma_S^I\mathrm{d}W_I\right]\\&-\frac{1}{2}\left[\left(\left(\sigma_S^S\right)^2+\left(\sigma_S^I\right)^2\right)+\sigma_S^S\sigma_S^I\rho_{SI}\right]t\Bigg]\end{aligned}\tag{A.5}$$

B. Proof of Proposition 3

Proof. Let

$$f\left(t,B\left(t,I\left(t\right)\right)\right)=\log B\left(t,I\left(t\right)\right).\tag{B.1}$$

Using Ito's lemma, we have that

$$\begin{aligned}\mathrm{d}\left(\log B\left(t,I\left(t\right)\right)\right)=&\left[\left(r_R\left(t\right)+\sigma_I\theta_I\right)B\left(t,I\left(t\right)\right)\mathrm{d}t+\sigma_I B\left(t,I\left(t\right)\right)\mathrm{d}W_I\left(t\right)\right]\\&+\frac{1}{2}\left[\frac{-1}{B\left(t,I\left(t\right)\right)}\right]\left[\sigma_I\mathrm{d}\left(t,I\left(t\right)\right)\right]^2.\end{aligned}\tag{B.2}$$

Integrating both sides of the above equation over the interval $\left[0,t\right]$ and taking expectation both sides we get:

$$\mathrm{d}\left(\log B\left(t,I\left(t\right)\right)\right)=B\left(0,I\left(0\right)\right)\exp\left[\int_0^t r_R\left(s\right)\mathrm{d}s+\left(\sigma_I\theta_I-\frac{1}{2}\left[\sigma_I\right]^2\right)t+\sigma_I\mathrm{d}W_I\right],\tag{B.3}$$

where $B(0, I(0))$ is the initial condition.

C. Proof of Theorem4

Proof. From Equation (2.16)

$$0 = \sup_{(u_S(t),u_B(t),c(t),y(t))\in\mathcal{A}} V_t(x,t) - \lambda V(x,t)$$
$$+V_x(s,X(s))\Big(\big\{r_R(t)+u_S\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)+u_B\sigma_I\theta_I\big\}X(t)-c(t)-y(t)\Big)$$
$$+\frac{1}{2}V_{xx}(s,X(s))\left(m_1(u_S)^2+m_2u_Su_B+m_3u_B^2\right)X^2+\lambda B(Z(t))-U(c,t).$$

We take the derivatives with respect to $u_B u_S, c$ and y to get

$$0=V_x(x,t)X(t)\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)+\frac{1}{2}V_{xx}(x,t)X^2\left(2m_1(u_S)+m_2u_B\right), \quad \text{(C.1)}$$

$$0=V_x(x,t)X(t)\sigma_I\theta_I+\frac{1}{2}V_{xx}(x,t)X^2\left(m_2(u_S)+2m_3u_B\right), \quad \text{(C.2)}$$

$$0=-V_x(x,t)+\mathrm{e}^{-\rho t}c^{\gamma-1}, \quad \text{(C.3)}$$

$$0=-V_x(x,t)+\frac{\lambda(t)\mathrm{e}^{-\rho t}}{\eta(t)}+\left(x+\frac{y(t)}{\mu(t)}\right)^{\gamma-1} \quad \text{(C.3)}$$

Solving for the optimals we get the following

$$u_B^*=\frac{1}{4m_1m_3-m_2^2}\left(2\left(\lambda_1\sigma_S+\lambda_2\sigma_I\theta_I\right)m_2+4\sigma_I\theta_Im_1\right)\frac{V_x}{xV_{xx}}, \quad \text{(C.4)}$$

$$u_S^*=\left[-\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)\left(1+\frac{m_2^2}{m_1\left(4m_1m_3-m_2^2\right)}\right)+\frac{2\sigma_I\theta_Im_1}{m_1\left(4m_1m_3-m_2^2\right)}\right]\frac{V_x}{xV_{xx}}, \quad \text{(C.5)}$$

$$c^*(t)=\left(\frac{1}{V_x(x,t)\mathrm{e}^{-\rho t}}\right)^{\frac{1}{1-\gamma}}, \quad \text{(C.6)}$$

$$x+\frac{y^*(t)}{\eta(t)}=\left(\frac{1}{V_x\mathrm{e}^{\rho t}}\frac{y(t)}{\mu(t)}\right)^{\frac{1}{1-\gamma}}. \quad \text{(C.7)}$$

Define the guess solution as

$$V(x,t)=\left(\frac{\mathrm{e}^{-\rho t}a(t)}{\gamma}\right)(x+b(t))^\gamma. \quad \text{(C.8)}$$

Taking partial derivatives we get,

$$V_t(x,t)=\mathrm{e}^{-\rho t}a(t)(x+b(t))^{\gamma-1}b'(t)+\left(\frac{-\mathrm{e}^{-\rho t}a(t)}{\gamma}\right)(x+b(t))^\gamma$$
$$+\left(\frac{\mathrm{e}^{-\rho t}a'(t)}{\gamma}\right)(x+b(t))^\gamma,$$

$$V_x(x,t)=\mathrm{e}^{-\rho t}a(t)(x+b(t))^{\gamma-1},$$

$$V_{xx}(x,t)=(\gamma-1)\mathrm{e}^{-\rho t}a(t)(x+b(t))^{\gamma-2}.$$

We substitute them into the Equation (2.16) to get

$$\begin{aligned}0=&\frac{\mathrm{e}^{-\rho t}}{\gamma}(x+b(t))^{\gamma}\left[\left(-\lambda(t)-\rho+\gamma\frac{G_1}{\gamma-1}\right)a(t)+\gamma\mathrm{e}^{\frac{\rho t}{\gamma-1}}a(t)^{\frac{1}{\gamma-1}}G_2+a'(t)\right]\\&+\mathrm{e}^{-\rho t}a(t)(x+b(t))^{\gamma-1}\left[b'(t)+x(r_R+\mu)\right]\end{aligned}, \quad (C.9)$$

with

$$G_1=C_1\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)+C_0\sigma_I\theta_I+\frac{1}{2}\left(m_1C_1^2+m_2C_1C_0+m_3C_0^2\right),$$

$$G_2=\mathrm{e}^{\frac{-\rho t}{1-\gamma}}\left[\left(\frac{1}{\gamma}-1\right)\lambda(t)^{\frac{1}{1-\gamma}}\mu^{\frac{-\gamma}{1-\gamma}}+1\right],$$

where

$$C_0=\frac{1}{4m_1m_3-m_2^2}\left(2\left(\lambda_1\sigma_S+\lambda_2\sigma_I\theta_I\right)m_2+4\sigma_I\theta_Im_1\right) \quad (C.10)$$

$$C_1=-\left(\lambda_1\sigma_S^S+\lambda_2\sigma_S^I\theta_I\right)\left(1+\frac{m_2^2}{m_1\left(4m_1m_3-m_2^2\right)}\right)+\frac{2\sigma_I\theta_Im_1}{m_1\left(4m_1m_3-m_2^2\right)} \quad (C.11)$$

Then we find the expressions for $a(t)$ and $b(t)$ to be

$$a(t)=\frac{-\gamma^2}{\gamma-1}\int_t^T\left[\exp\left(\frac{-\gamma}{\gamma-1}\int_t^s\left(-\lambda(v)-\rho+\frac{-\gamma}{\gamma-1}G_1\right)\mathrm{d}v\right)\exp\left(\frac{\rho}{\gamma-1}s\right)G_2\right]\mathrm{d}s \quad (C.12)$$

$$b(t)=\int_t^T-\left(y(s)+x(r_R+\mu)\right)\mathrm{d}s. \quad (C.13)$$

Thus optimal amounts spent on stock, bond, consumption and income are given by

$$u_B^*=\frac{x+b(t)}{x(\gamma-1)}C_0 \quad (C.14)$$

$$u_S^*=\frac{x+b(t)}{x(\gamma-1)}C_1 \quad (C.15)$$

$$c^*(t)=a(t)^{\frac{1}{\gamma-1}}(x+b(t)) \quad (C.16)$$

$$y^*(t)=\eta(t)\left\{\left(\frac{\lambda(t)}{\eta(t)}\right)^{\frac{1}{1-\gamma}}a(t)^{\frac{1}{\gamma-1}}(x+b(t))-x\right\} \quad (C.17)$$