Risk Exchange under EUUP

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Abstract

In this paper, we consider an equilibrium insurance premium and risk exchange in a pure exchange economy with ambiguity or Knightian uncertainty. Each agent's preference is represented by the expected utility with uncertainty probability (EUUP) theory. The Bühlmann's economic premium principle is generalized under EUUP. Contrary to the existing models, our principle under uncertainty is given unanimously and can be calculated more easily and explicitly. Through comparative statics, we show that insurance transactions occur and demand for insurance is not always comonotonic due to the difference in the degree of ambiguity aversion even if all of the agents in the economy are ambiguity averse or ambiguity loving.

Keywords

Ambiguity, Economic Premium Principle, Equilibrium, Expected Utility with Uncertainty Probability (EUUP), Risk Exchange

1. Introduction

Equilibrium asset pricing models of financial and insurance markets have been extensively studied in the literature. Following the pioneering work of [1] in the field, [2] [3] established *the economic premium principle*, which is an equilibrium insurance pricing model in a pure exchange economy under expected utility (EU) theory. Later, to overcome the violations of EU such as the famous Ellsberg paradox of [4], which says people avoid Knightian uncertainty or ambiguity[1], [5] generalized Bühlmann's economic premium principle under maxmin EU (MEU) of [6], a special version of Choquet EU (CEU) of [7][2]. However,

[1]Uncertainty is used as an umbrella term for both *risk* and *ambiguity*. *Risk* is defined as a condition in which odds of possible events are perfectly known. *Ambiguity* refers to a condition in which odds of possible events are either not uniquely assigned or not perfectly known.

[2][5] also derived an economic premium principle under rank-dependent EU (RDEU) of [8], because it formally coincides with the one under MEU.

their economic premium principle or the state price density (SPD) in equilibrium is generally indeterminate because they are derived under a reference measure that is not uniquely determined[3]. Recently, [9] developed an economic premium principle under the dual theory [10] of the smooth ambiguity model of [11]. They showed the uniqueness and existence of the SPD in equilibrium. Although the search for models of decision-making and insurance pricing under ambiguity has been evolving toward the ultimate separations between risk and ambiguity, the above models still maintain some relation between risk and ambiguity and the search for an applicable model is still ongoing.

This paper further develops a model that can be used in empirical and behavioral studies by refining the separations between tastes and beliefs, and between risk and ambiguity. To this end, we adopt a new decision-making model called *expected utility with uncertain probability* (EUUP) theory of [12]. EUUP can completely distinguish tastes from beliefs and risk from ambiguity. It identifies the sources of uncertainty (consequences versus probabilities), allowing attitude toward ambiguity to be completely elicited and characterized explicitly.

We consider a pure exchange economy where insurance is traded. The economy is identical with [3] except that economic agents face not risk but ambiguity and their preference is represented by EUUP. We show that the equilibrium exists and is unique in this economy[4]. Then, we derive the equilibrium insurance premium, which can be viewed as a generalization of Bühlmann's economic premium principle under EUUP. This economic premium principle coincides with Bühlmann's when agents are ambiguity neutral. We also study the effects of ambiguity on both insurance demand and premium through comparative statics analysis. Through this analysis, we numerically confirm that insurance transactions or risk exchanges occur due to the difference in the degree of ambiguity aversion even if all the agents are ambiguity averse or ambiguity loving as well as when they share the same attitude toward risk. We also show that transactions occur if some agents are ambiguity averse and others are ambiguity loving but all of them share the same attitude toward risk.

These results are our main contribution to existing literature. [5] [13] and [14] show that optimal insurance demand or risk sharing is comonotonic in equilibrium under CEU or MEU assuming that all agents in the economy are ambiguity averse. However, our results show that their assertion does not necessarily hold. Furthermore, although [9] shows sufficient conditions such that an increase in ambiguity aversion increases demand and premium for insurance against ambiguous loss, it is difficult to confirm whether this condition holds even if we specify the form of the utility function and make some parameterization, and it is also difficult to derive explicitly the numerical result. On the other hand, because we can explicitly derive results numerically in our model, it is much easier to apply our model to empirical studies and actual experiments.

This paper is organized as follows. We formally state our model in Section 2.

[3]The reference measure is a probability measure under which the expected utility attains MEU.

[4][5] did not show the existence and uniqueness of equilibrium.

In Section 3, we derive our economic premium principle. Based on the equilibrium insurance premium, we perform some comparative statics in Section 4. Finally, we make concluding remarks in Section 5.

2. The Market Model

We consider a single period pure exchange economy that consists of n agents (each agent is denoted as agent i, $i=1,2,\cdots,n$). The commodities to be traded are quantities of money, conditional on the state $s\in S\subset\mathbb{R}$ with a Borel measurable space $(S,\mathcal{B}(S))$[5]. Let $X_i(s)$ denote the loss agent i faces, at the terminal time of the period if the state $s\in S$ occurs. At the start of the period, to hedge the loss, each agent buys or sells an insurance policy that pays $Y_i(s)$ at the terminal time of the period if the state s occurs. Let $\mathbb{P}=\{P_\theta(s);s\in S\}_{\theta\in\Theta}$ be a set of probability measures on $(S,\mathcal{B}(S))$, where $\Theta\subset\mathbb{R}$ is an interval in the real line that stands for an arbitrary index set. Ambiguity is represented by a probability measure Q on $(\Theta,\mathcal{B}(\Theta))$, which is assumed to be the same among all agents[6].

Agent i is characterized by her/his utility function $u_i:\mathbb{R}\to\mathbb{R}$. The utility function u_i is assumed to be strictly increasing, strictly concave and twice continuously differentiable with the properties $\lim_{x\to\infty}u_i'(x)=0$ and $\lim_{x\to-\infty}u_i'(x)=\infty$.

For each $s\in S$, assuming that $P_\theta(s)$ is $\mathcal{B}(\Theta)$-measurable, we define a probability measure on $(S,\mathcal{B}(S))$ by $\bar{P}(s)=\int_{\theta\in\Theta}P_\theta(s)\mathrm{d}Q(\theta)$. $\bar{P}$ is the probability measure when agents are ambiguity neutral. We call the probability measure $\bar{P}$ the *reference measure*[7] because this probability measure is known to every agent. Let $\mathbb{E}$ be the expectation under $\bar{P}$. We also assume that insurance traded in the market is priced through $\bar{P}$, that is, $Y_i=\{Y_i(s):s\in S\}$ can be bought or sold by agent i at a price;

$$p(Y_i)=\mathbb{E}[Y_i\phi]=\int_{s\in S}Y_i(s)\phi(s)\mathrm{d}\bar{P}(s),$$

where $\phi:S\to(0,\infty)$ is the *state price density* (SPD) which satisfies $\mathbb{E}[\phi]=1$. We note that although the expectation is taken under the reference measure, agents' ambiguity attitudes (ambiguity averse, ambiguity neutral, or ambiguity seeking) are reflected in the SPD as shown later.

Let w_i be the initial wealth of agent i, then, for a given (w_i,X_i,Y_i), the terminal wealth $W_i=\{W_i(s):s\in S\}$ of agent i is given by

$$W_i(s)=w_i-X_i(s)+Y_i(s)-p(Y_i),\quad s\in S.\text{[8]}\qquad(1)$$

Then, assuming that each agent evaluates her terminal wealth by the EUUP

[5] $\mathcal{B}(S)$ denotes the Borel σ-field of S.

[6] Because preferences for the probability measure Q can be different among agents as seen later, this assumption seems not unrealistic.

[7] Here and hereafter, the term *reference measure* is used differently in the context of the MEU, but it should not lead to confusion.

[8] Equalities and inequalities for random variables hold in the sense of a.e.; however, we omit the notation a.e. for the sake of notational simplicity.

without distortion of perceived probabilities[9], the welfare of agent *i* is given by

$$V_i(W_i)=\int_{z\le 0}\left(G^{(i)}_{u_i(W_i)}(z)-1\right)\mathrm{d}z+\int_{z\ge 0}G^{(i)}_{u_i(W_i)}(z)\mathrm{d}z, \tag{2}$$

where $G^{(i)}_{u_i(W_i)}$ denotes a capacity on $(S,\mathcal{B}(S))$[10], that is defined by

$$\begin{aligned}G^{(i)}_{u_i(W_i)}(z)&=\varphi_i^{-1}\left(\int_{\theta\in\Theta}\varphi_i\left(P_\theta\{s\in S:u_i(W_i(s))>z\}\right)\mathrm{d}Q(\theta)\right)\\&=\varphi_i^{-1}\left(\int_{\theta\in\Theta}\varphi_i\left(1-F_{W_i}\left(u_i^{-1}(z);\theta\right)\right)\mathrm{d}Q(\theta)\right),\quad z\in\mathbb{R},\end{aligned}$$

where $\varphi_i:[0,1]\to[0,1]$ is a strictly increasing continuous function that represents agent *i*'s attitude toward ambiguity and that is referred to as the *probability outlook function*, and where, for each $\theta\in\Theta$, $F_{W_i}(\cdot;\theta)$ denotes the cumulative distribution function (cdf) of W_i under the probability measure P_θ. We note that the valuation function (2) coincides with the cumulative prospect theory (CPT) of [15] if either θ or Q is singleton. Furthermore, if agents' preferences for risk and ambiguity are sign-independent, it coincides with the CEU of [7]. Hence, it is considered as an extension of them to the multiprior model advocated by [16].

We consider a problem in which each agent *i* decides the amount Y_i to maximize the welfare $V_i(W_i)$. More precisely, agent *i* faces the following maximization problem;

$$\left|\begin{array}{ll}\text{Maximize} & V_i(W_i)\\ \text{s. to} & w_i=\mathbb{E}\left[\phi(W_i+X_i)\right].\end{array}\right. \tag{3}$$

Once an optimal terminal wealth W_i^* that solves (3) is given, agent *i*'s optimal insurance Y^* is given by

$$Y_i^*-p(Y_i^*)=W_i^*+X_i-w_i. \tag{4}$$

Here, we note that the optimal insurance Y_i^* is only determined up to an additive constant because, for any constant *c*,
$Y_i^*-c-\mathbb{E}\left[\phi\left(Y_i^*-c\right)\right]=Y_i^*-\mathbb{E}\left[\phi Y_i^*\right]$. Hence, we normalize the optimal insurance Y_i^* so as to satisfy $\mathbb{E}\left[\phi Y_i^*\right]=0$ hereafter.

To solve the maximization problem (3), we first represent the terminal welfare $V_i(W_i)$ as the expected utility under agent *i*'s perceive probability measure as shown in the following lemma.

Lemma 1 *Let* $\hat{F}_{i,W_i}:\mathbb{R}\to[0,1]$ *be agent i's perceived-cumulative distribution function* (*perceived-cdf*) *of* W_i, *defined by*

[9]The original EUUP of [12], instead of (2), is given by

$$V_i(W_i)=\int_{z\le 0}\left[\Gamma\left(G^{(i)}_{u_i(W_i)}(z)\right)-1\right]\mathrm{d}z+\int_{z\ge 0}\Gamma\left(G^{(i)}_{u_i(W_i)}(z)\right)\mathrm{d}z,$$

where Γ denotes "decision weight" that distorts perceived probabilities. However, because including Γ leads to the analysis becoming extremely difficult, we omit it in the paper. Considering the effects of Γ in the sequel analysis remains a future task.

[10]A *capacity* on $(S,\mathcal{B}(S))$ is a set function $G:S\to[0,1]$ satisfying $G(A)\le G(B)$ for any $A,B\in\mathcal{B}(S)$ such that $A\subset B$, and $G(S)=1$. That is, it is a function such that the requirement for additivity is removed from a probability measure.

$$\hat{F}_{i,W_i}(x) := \bar{\varphi}_i^{-1}\left(\int_{\theta\in\Theta}\bar{\varphi}_i\left(F_{W_i}(x;\theta)\right)\mathrm{d}Q(\theta)\right), \quad x\in\mathbb{R}, \tag{5}$$

where $\bar{\varphi}_i$ be the dual of the outlook function φ_i, that is defined by

$$\bar{\varphi}_i(p) := 1-\varphi_i(1-p), \quad p\in[0,1].$$

Then, the value function V_i can be rewritten as

$$V_i(W_i) = \int_{x\in\mathbb{R}} u_i(x)\mathrm{d}\hat{F}_{i,W_i}(x). \tag{6}$$

In other words, let $\hat{P}_i$ be the induced probability measure induced by the perceived-cdf $\hat{F}_{i,W_i}$. Then the value function V_i is given by the expected utility under $\hat{P}_i$ as

$$V_i(W_i) = \hat{\mathbb{E}}_i\left[u_i(W_i)\right],$$

where $\hat{\mathbb{E}}_i$ denotes expectation under $\hat{P}_i$.

Proof. We first note that using the dual of the outlook function, we can rewrite $G^{(i)}_{u_i(W_i)}$ as

$$\begin{aligned} G^{(i)}_{u_i(W_i)}(x) &= 1-\bar{\varphi}_i^{-1}\left(\int_{\theta\in\Theta}\bar{\varphi}_i\left(F_{i,W_i}\left(u^{-1}(x);\theta\right)\right)\mathrm{d}Q(\theta)\right) \\ &= 1-\hat{F}_{i,W_i}\left(u^{-1}(x)\right), \quad x\in\mathbb{R}. \end{aligned}$$

By applying the change of variable

$$p = \hat{F}_{i,W_i}\left(u^{-1}(z)\right) \Leftrightarrow z = u_i\left(\hat{F}^{-1}_{i,W_i}(p)\right) \tag{7}$$

to (2), and performing integration by parts, we obtain

$$\begin{aligned} V_i(W_i) &= -\int_0^{\hat{F}_{i,W_i}\left(u_i^{-1}(0)\right)} p\mathrm{d}u_i\left(\hat{F}^{-1}_{i,W_i}(p)\right) + \int_{\hat{F}_{i,W_i}\left(u_i^{-1}(0)\right)}^1 (1-p)\mathrm{d}u_i\left(\hat{F}^{-1}_{i,W_i}(p)\right) \\ &= \int_0^1 u_i\left(\hat{F}^{-1}_{i,W_i}(p)\right)\mathrm{d}p. \end{aligned}$$

Applying the change of variable

$$x = \hat{F}^{-1}_{i,W_i}(p)$$

to the above equation leads to

$$V_i(W_i) = \int_{x\in\mathbb{R}} u_i(x)\mathrm{d}\hat{F}_{i,W_i}(x).$$

Hence, we obtain the result.

From the above lemma, agent *i*'s welfare maximization problem can be rewritten as follows.

$$\left|\begin{array}{ll} \text{Maximize} & \hat{\mathbb{E}}_i\left[u_i(W_i)\right] \\ \text{s. to} & w_i = \mathbb{E}\left[\phi(W_i + X_i)\right]. \end{array}\right. \tag{8}$$

That is, we can formulate the problem as a parallel problem to the classical expected utility maximization problem as can be seen in (8). We can solve this problem by an orthodox method. The results are shown in the following proposition.

Proposition 1 *Let* I_i *be the inverse function of the marginal utility* u_i', *and let* L_i *be the Radon-Nikodym derivative of* $\hat{P}_i$ *with respect to* $\bar{P}$, *i.e.*,

$$L_i = \frac{\mathrm{d}\hat{P}_i}{\mathrm{d}\bar{P}}. \tag{9}$$

Then, agent *i*'s optimal terminal wealth W_i^* is given by

$$W_i^* = I_i\left(\lambda_i L_i^{-1}\phi\right), \tag{10}$$

where λ_i is a positive constant defined by

$$w_i - \mathbb{E}\left[\phi X_i\right] = \mathbb{E}\left[\phi I_i\left(\lambda_i L_i^{-1}\phi\right)\right]. \tag{11}$$

Furthermore, agent *i*'s optimal insurance Y_i^* is given by

$$Y_i^* = I_i\left(\lambda_i L_i^{-1}\phi\right) + X_i(s) - w_i. \tag{12}$$

Proof. From (9), we can rewrite (8) as

$$\left|\begin{array}{ll} \text{Maximize} & \hat{\mathbb{E}}_i\left[u_i\left(W_i\right)\right] \\ \text{s. to} & w_i = \hat{\mathbb{E}}\left[\phi\left(W_i + X_i\right)L_i\right]. \end{array}\right.$$

This is an orthodox concave maximization problem with a linear constraint. Hence we can immediately obtain the results of (10) and (11) with the Lagrange multiplier λ_i (See e.g., Theorem 6.3 of [17]). Finally, we have (12) from (14).

Hence, (12) in Proposition 1, which gives the optimal insurance, is rewritten as

$$Y_i^* = I_i\left(\lambda_i L_i^{-1}\phi\right) + X_i - w_i. \tag{13}$$

3. Equilibrium and the State Price Density

We define an equilibrium in the economy described in the previous section as follows.

Definition 1 $\left(\phi, Y_1^*, \cdots, Y_n^*\right)$ *is an equilibrium if the following two conditions are satisfied.*

For each $i = 1, \cdots, n$, Y_i^* is the Agent *i*'s optimal insurance policy given by Proposition 1;

$$\sum_{i=1}^{n} I_i\left(\lambda_i L_i^{-1}(s)\phi(s)\right) = w - X(s), \quad s \in S, [11] \tag{14}$$

where $w := \sum_{i=1}^{n} w_i$ and $X(s) := \sum_{i=1}^{n} X_i(s)$ are the aggregate initial wealth and the aggregate loss, respectively.

For each $s \in S$, let $\mathcal{I}(x; s, \boldsymbol{\lambda})$ be defined by

$$\mathcal{I}(x; s, \boldsymbol{\lambda}) := \sum_{i=1}^{n} I_i\left(\frac{\lambda_i}{L_i(s)} x\right).$$

Then, (14) is rewritten as

$$\mathcal{I}\left(\phi(s); s, \boldsymbol{\lambda}\right) = w - X(s), \quad s \in S.$$

Because $\mathcal{I}(x; s, \boldsymbol{\lambda})$ is trivially a strictly decreasing function, it has its inverse

[11](14) is equivalent to $\sum_{i=1}^{n} Y_i^*(s) = 0$. This means we consider exchange problems among the agents in the equilibrium.

function. Hence, if the inverse function $\mathcal{H}(\cdot;s,\boldsymbol{\lambda})$ of $\mathcal{I}(\cdot;s,\boldsymbol{\lambda})$ is defined by

$$\mathcal{I}\left(\mathcal{H}\left(x;s,\boldsymbol{\lambda}\right);s,\boldsymbol{\lambda}\right)=x,\quad s\in S, \tag{15}$$

then the SPD in equilibrium is given by

$$\phi(s)=\mathcal{H}\left(w-X\left(s\right);s,\boldsymbol{\lambda}\right),\quad s\in S.$$

Because in the equilibrium the budget constraint (11) can be rewritten as

$$\mathbb{E}\left[\mathcal{H}\left(w-X;\boldsymbol{\lambda}\right)\left(I_i\left(\frac{\lambda_i}{L_i}\mathcal{H}\left(w-X;\boldsymbol{\lambda}\right)\right)+X_i\right)\right]=w_i, \tag{16}$$

the equilibrium can be characterized by $\boldsymbol{\lambda}=\left(\lambda_1,\cdots,\lambda_n\right)\in\left(0,\infty\right)^n$ satisfying (16).

The following proposition states the existence and uniqueness of the equilibrium.

Proposition 2 *There exists a* $\boldsymbol{\lambda}\in\left(0,\infty\right)^n$ *satisfying* (16). *Furthermore, suppose that, for each agent* i, $i=1,\cdots,n$, *the Arrow-Pratt relative risk aversion satisfies the condition*:

$$-x\frac{u_i''(x)}{u_i'(x)}\le 1,\quad x\in\left(0,\infty\right). \tag{17}$$

Then, $\boldsymbol{\lambda}$ is unique up to positive constant multiples.

Proof. By the same arguments as the proof of Theorem 4.6.1 of [17], we obtain the proposition.

We note that the restriction (17) on relative risk aversion is important in the consumption-saving problem to sign the comparative statics of (stochastic) changes in the interest rate, see [18] and [19].

Solution for Exponential Utility

When we specify the form of utility function as log or exponential functions, we can derive the SPD in equilibrium analytically as a closed-form function. In this subsection, assuming that each agent has an exponential utility function, we derive the SPD and optimal demands for insurance explicitly.

Proposition 3 *Suppose that each agent have an exponential utility function with an index of constant risk aversion* $\rho_i>0$, *that is*,

$$u_i\left(x\right)=\frac{1}{\rho_i}\left(1-\mathrm{e}^{-\rho_i x}\right),\quad i=1,\cdots,n. \tag{18}$$

Then, the SPD is given in equilibrium as

$$\phi=\frac{\mathrm{e}^{\bar{\rho}X}\prod_{i=1}^{n}L_i^{\frac{\bar{\rho}}{\rho_i}}}{\mathbb{E}\left[\mathrm{e}^{\bar{\rho}X}\prod_{i=1}^{n}L_i^{\frac{\bar{\rho}}{\rho_i}}\right]}. \tag{19}$$

where $\bar{\rho}$ is a constant defined by $\frac{1}{\bar{\rho}}=\sum_{i=1}^{n}\frac{1}{\rho_i}$.

Proof. Because $u_i'(x) = \mathrm{e}^{-\rho_i x}$ and $I_i(y) = -\frac{1}{\rho_i}\log y$ in this case, the market clearing condition (14) is given as

$$\begin{aligned} w - X &= \sum_{i=1}^{n} I_i\left(\frac{\lambda_i}{L_i}\phi\right) \\ &= -\sum_{i=1}^{n}\frac{1}{\rho_i}\log\left(\frac{\lambda_i}{L_i}\phi\right) \\ &= -\sum_{i=1}^{n}\frac{1}{\rho_i}\log\left(\frac{\lambda_i}{L_i}\right) - \frac{1}{\bar{\rho}}\log\phi. \end{aligned}$$

From this, we have

$$\log\phi = \bar{\rho}X + \log\prod_{i=1}^{n} L_i^{\frac{\bar{\rho}}{\rho_i}} - \bar{\rho}w - \log\prod_{i=1}^{n}\lambda_i^{\frac{\bar{\rho}}{\rho_i}}$$

or $\phi = \mathrm{e}^{\bar{\rho}X}\prod_{i=1}^{n} L_i^{\frac{\bar{\rho}}{\rho_i}}\kappa,$

where κ is a constant defined by $\kappa = \mathrm{e}^{-\bar{\rho}w}\prod_{i=1}^{n}\lambda_i^{-\frac{\bar{\rho}}{\rho_i}}$. On the other hand, from the definition of the SPD, because $\mathbb{E}[\phi] = 1$, κ must coincide with

$$\frac{1}{\mathbb{E}\left[\mathrm{e}^{\bar{\rho}X}\prod_{i=1}^{n} L_i^{\frac{\bar{\rho}}{\rho_i}}\right]}.$$

Hence, we obtain the result.

We note that if $L_i = 1$, $i = 1,\cdots,n$, the SPD (19) coincides with that of [2] or the Esscher transform. If agent *i* is ambiguity neutral, the probability outlook function φ_i is given as $\varphi_i(x) = x$, $x \in [0,1]$ (Theorem 2 of [12]). This leads to $L_i = 1$ because $\hat{P}_i = \bar{P}$ in this case.

Next, we derive the optimal insurance for each agent.

Proposition 4 *Under the same assumption of Proposition* 3, *the agent i's optimal insurance* Y^* *is given as*

$$\begin{aligned} Y_i^* &= -\frac{\bar{\rho}}{\rho_i}\left(X - \mathbb{E}[\phi X]\right) - \frac{\bar{\rho}}{\rho_i}\sum_{j\neq i}\frac{1}{\rho_j}\left(\log\frac{L_j}{L_i} - \mathbb{E}\left[\phi\log\frac{L_j}{L_i}\right]\right) \\ &\quad + X_i - \mathbb{E}[\phi X_i], \quad s \in S. \end{aligned} \tag{20}$$

Proof. Because $I_i(y) = -\frac{1}{\rho_i}\log y$ by (18), from (13), we have

$$\begin{aligned} Y_i^* &= I_i\left(\frac{\lambda_i\phi}{L_i}\right) + X_i - w_i \\ &= -\frac{1}{\rho_i}\log\lambda_i - \frac{1}{\rho_i}\log\phi + \frac{1}{\rho_i}\log L_i + X_i - w_i. \end{aligned} \tag{21}$$

Multiplying by ϕ on both sides of the above equations, and taking the expectation $\mathbb{E}$ leads to

$$0 = -\frac{1}{\rho_i}\log\lambda_i - \frac{1}{\rho_i}\mathbb{E}[\phi\log\phi] + \frac{1}{\rho_i}\mathbb{E}[\phi\log L_i] + \mathbb{E}[\phi X_i] - w_i. \tag{22}$$

Here we note that we have used the fact that $\mathbb{E}\left[\phi Y^*\right]=0$ and $\mathbb{E}[\phi]=1$. Cancelling out $-\frac{1}{\rho_i}\log\lambda_i$ from (21) and (22), we have

$$Y_i^* = -\frac{1}{\rho_i}\left(\log\phi - \mathbb{E}[\phi\log\phi]\right) + \frac{1}{\rho_i}\left(\log L_i - \mathbb{E}[\phi\log L_i]\right) + X_i - \mathbb{E}[\phi X_i].$$

Finally, substituting (19) into the above equation, we obtain (20).

We also note that if $L_i \equiv 1$, $i=1,\cdots,n$, optimal insurance Y^* in (20) coincides with that of [2]'s model. (20) shows the optimal insurance or risk exchange decomposes into *individual risk* $X_i - \mathbb{E}[\phi X_i]$, *market risk* $-\frac{\bar{\rho}}{\rho_i}\left(X - \mathbb{E}[\phi X]\right)$, and *difference of attitude toward ambiguity* $-\frac{\bar{\rho}}{\rho_i}\sum_{j\neq i}\frac{1}{\rho_j}\left(\log\frac{L_j}{L_i} - \mathbb{E}\left[\phi\log\frac{L_j}{L_i}\right]\right)$.

4. Numerical Examples and Comparative Statics

In this section, we make some comparative statics to examine the effects of ambiguity on insurance demand and premium under the assumption that all agents have exponential utility functions treated in the previous section.

We consider the economy consisting of two agents $i=1,2$. To eliminate other effects and focus only on the effect of ambiguity, we make the following assumptions. First, each agent faces the same amount of loss $X_i(s) = -e^{-s}$ at the terminal time of the period, that is, the amounts of loss are strictly decreasing w.r.t. state $s\in\mathbb{R}$ [12]. Second, we assume that each agent shares the same type exponential utility function such that $u_i(x) = \frac{1-e^{-\rho x}}{\rho}$, $i=1,2$. That is, each agent possesses the same constant index of risk aversion ρ. In other words, each agent shares the same attitude toward risk. In the sequel, we specify the value of the index of risk aversion as $\rho = 0.00064$ [13]. As to the ambiguity, the probability distribution of the state $s\in\mathbb{R}$ follows the normal distribution with mean $\theta\in\{-1,0,1\}$, and standard deviation 1. That is, the probability density function of the state is given by

$$f(x;\theta) = \begin{cases} f(x;-1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x+1)^2}{2}} & \text{w.p. } \frac{1}{3} \\ f(x;0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} & \text{w.p. } \frac{1}{3} \\ f(x;1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-1)^2}{2}} & \text{w.p. } \frac{1}{3} \end{cases}$$

[12]Because we assume that the terminal wealth of each agent is strictly increasing w.r.t. $s\in R$, the assumption such that the amount of loss is strictly decreasing w.r.t. $s\in\mathbb{R}$ seems to be reasonable.
[13]Because [20] find 0.00063 or 0.00064 for absolute risk aversion of CARA utility based on real-world insurance choices, we adopt 0.00064 as this value.

We assume that the function that represents agent i's attitude toward ambiguity follows CAAA (Constant Absolute Ambiguity Aversion) type; such that

$$\varphi_i(x) = \frac{1-e^{-\tau_i x}}{\tau_i},$$

where $\tau_i = -\frac{\varphi_i''(x)}{\varphi_i'(x)}$ is a positive constant that represents her/his degree of ambiguity aversion. We note that if $\tau_i > 0$ then agent i is ambiguity averse, and otherwise if $\tau_i < 0$ then she/he is ambiguity loving, and that if $\tau_1 > (<) \tau_2$ then agent 1 is more ambiguity averse (loving) than agent 2 (see [12]).

First, we show the behavior of the likelihood ratio $L_i(s)$. **Figure 1** is a comparison of $L_i(s)$ among the agents with different coefficients of ambiguity aversion $\tau_i = -1.4, -0.7, 0.7, 1.4$ [14]. Here and hereafter, in all the graphs, the horizontal axis denotes the state $s \in \mathbb{R}$. Because it is assumed that the value of the state $s \in \mathbb{R}$ is larger, the terminal wealth $W_i(s)$ of each agent is larger, the case where the value of the state is large is referred to as a "good state," and the case where the value of the state is low is referred to as a "bad state." From **Figure 1**, if an agent is ambiguity averse, *i.e.*, $\tau_i = 0.7$ or $\tau_i = 1.4$, she/he overestimates the value of the capacity against the corresponding reference probability measure in the bad states and underestimates it in the good states. The more ambiguity averse she/he is, the larger the difference between the capacity and the reference probability measure is.

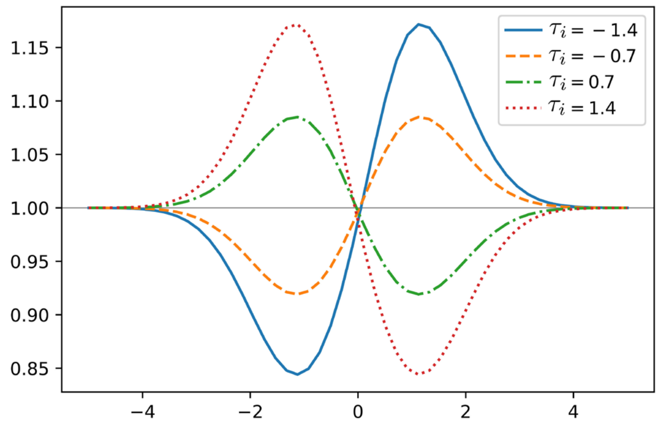

Figure 1. The likelihood ratio $L_i(s)$. This figure shows a comparison of the likelihood ratio $L_i(s) = \frac{\hat{P}_i(s)}{\bar{P}(s)}$ among the agents with different coefficients of ambiguity aversion $\tau_i = -1.4, -0.7, 0.7, 1.4$. Here and hereafter, in all the graphs, the horizontal axis denotes the state $s \in \mathbb{R}$.

[14][21], assuming the representative investor is the EUUP maximizer, empirically estimates the coefficient of ambiguity aversion of the representative investor in the US stock market as from −1.446 to 1.829. Hence, we set $\tau_i = -1.4, -0.7, 0.7, 1.4$ as the coefficients of ambiguity aversion in our numerical examples.

On the other hand, if an agent is ambiguity loving, *i.e.*, $\tau_i = -0.7$ or $\tau_i = -1.4$, the curve seems to be symmetric w.r.t. the horizontal line such that $L_i(s) = 1$. That is, the value of the capacity against the corresponding reference probability measure is lower in the bad states and it is higher in the good states. The more ambiguity loving the agent is, the larger the difference between the capacity and the reference probability measure.

From these results, we can see that if an agent is more ambiguity averse (loving), she perceives the likelihood of a bad state to be higher (lower) and that of a good state to be lower (higher).

Next, we compare the behavior of the SPD ϕ under the EUUP with that of ϕ_0 under the Bühlmann model. First, **Figure 2** shows the graph of the SPD ϕ_0 under the Bühlmann model. In the model, because the SPD is an exponential function of the aggregate risk, ϕ_0 decays exponentially as the value of the states increases. **Figure 3** shows the case where all the agents are ambiguity averse and the coefficients of the ambiguity aversion τ_i of the agents are 1.4 and 0.7. In this case, except for the extreme states such that $s < -3$ or $s > 3$, the SPD ϕ compared with the SPD ϕ_0 is higher in the bad states, and it is lower in the good states. On the other hand, **Figure 4** shows the case where all of agents are ambiguity loving and the coefficients of the ambiguity aversion τ_i of the agents are −1.4 and −0.7. In this case, opposite to the case where all the agents are ambiguity averse, the SPD ϕ compared with the SPD ϕ_0 is lower in the bad states, and it is higher in the good states except for the extreme states. **Figure 5** shows the case where one agent is ambiguity averse and the other is ambiguity loving and their coefficients of the ambiguity aversion are −1.4 and 1.4, respectively. In this case, the result is a mixture of that of the case where all the agents are ambiguity averse and that of the case where all the agents are ambiguity loving. In all cases, the increase in ϕ in the extremely bad states is modest compared with ϕ_0.

Finally, we investigate insurance demand in the equilibrium. **Figure 6** shows insurance demand in the economy where both agents are ambiguity averse, **Figure 7** shows it in the economy where both of agents are ambiguity loving, and **Figure 8** shows it in the economy where one of the agents is ambiguity loving and the other agent is ambiguity averse.

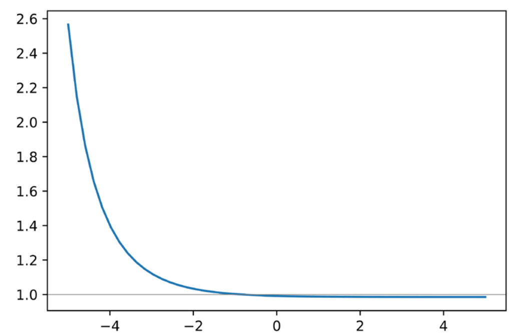

Figure 2. The SPD ϕ_0. This figure shows the graph of the SPD ϕ_0 under the Bühlmann model where each agent shares the same constant index of risk aversion $\rho_1 = \rho_2 = 0.00064$.

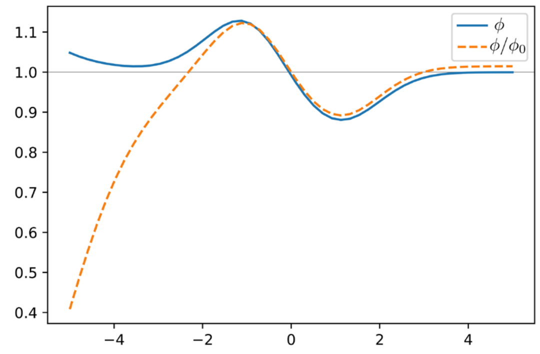

Figure 3. ϕ and ϕ/ϕ_0; Ambiguity Averters. $\tau_i = 1.4, 0.7$. This figure shows the SPD ϕ under the EUUP and its ratio to the SPD ϕ_0 under the Bühlmann model. Here all the agents are ambiguity averse and the coefficients of the ambiguity aversion τ_i of the agents are 1.4 and 0.7.

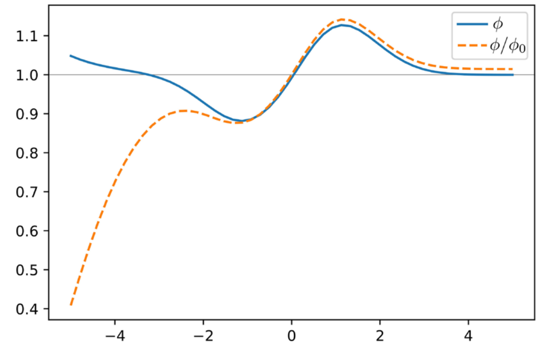

Figure 4. ϕ and ϕ/ϕ_0; Ambiguity Lovers. $\tau_i = -1.4, -0.7$. This figure shows the SPD ϕ under the EUUP and its ratio to the SPD ϕ_0 under the Bühlmann model. Here all of agents are ambiguity loving and the coefficients of the ambiguity aversion τ_i of the agents are −1.4 and −0.7.

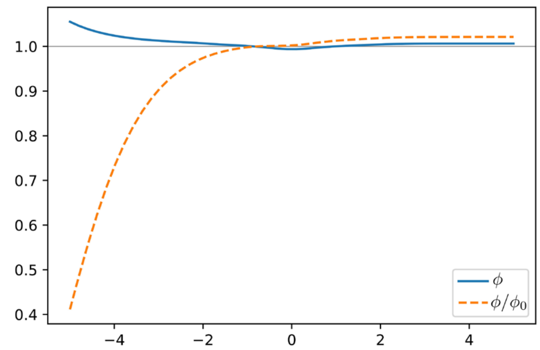

Figure 5. ϕ and ϕ/ϕ_0; Ambiguity Lover & Averter. $\tau_i = -1.4, 1.4$. This figure shows the SPD ϕ under the EUUP and its ratio to the SPD ϕ_0 under the Bühlmann model. Here one agent is ambiguity averse and the other is ambiguity loving and their coefficients of the ambiguity aversion are −1.4 and 1.4, respectively.

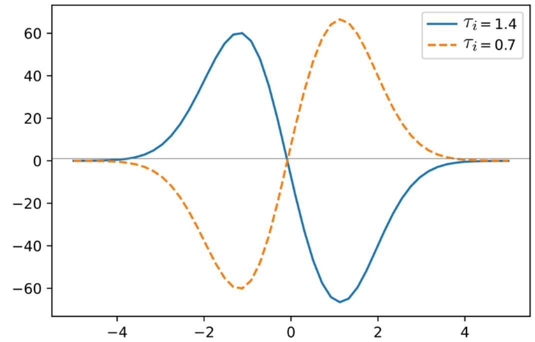

Figure 6. Y_i^*; Ambiguity Averters. $\tau_i = 1.4, 0.7$. This figure shows insurance demand in the economy where all the agents are ambiguity averse and the coefficients of the ambiguity aversion τ_i of the agents are 1.4 and 0.7.

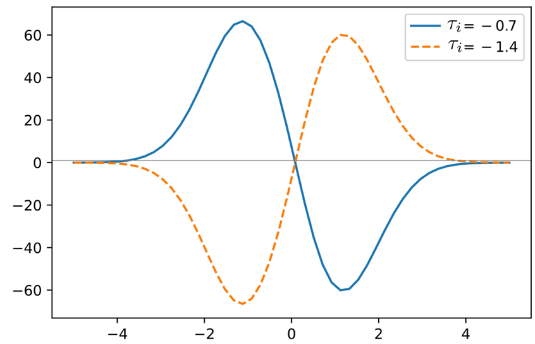

Figure 7. Y_i^*; Ambiguity Lovers. $\tau_i = -0.7, -1.4$. This figure shows insurance demand in the economy where all the agents are ambiguity loving and the coefficients of the ambiguity aversion τ_i of the agents are −0.7 and −1.4.

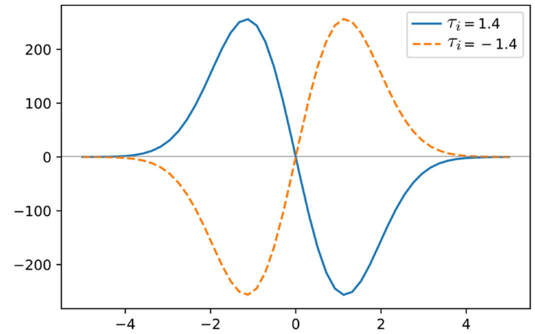

Figure 8. Y_i^*; Ambiguity Averter & Lover. $\tau_i = 1.4, -1.4$. This figure shows insurance demand in the economy where one of the agents is ambiguity loving and the other agent is ambiguity averse, and their coefficients of the ambiguity aversion are −1.4 and 1.4, respectively.

From Figure 6 and Figure 7, we note that insurance transactions occur due to the difference in the degree of ambiguity aversion even if all the agents are ambiguity averse or ambiguity loving. From Figures 6-8, we can see that the more ambiguity averse agent or the less ambiguity loving agent receives the insurance amount in the bad states, and she/he pays it in the good states.

In summary, from the above numerical examples, we can see more ambiguity averse agents overestimate (underestimate) the bad (good) states compared with the reference probability measure and receive (pay) the insurance amount in the good (bad) states.

5. Concluding Remarks

In this paper, we derive an equilibrium insurance premium and risk exchange in a pure exchange economy with ambiguity, where agents follow EUUP theory of [12]. Our premium principle is an extension of the economic premium principle by [2] under the EUUP. Applying the EUUP theory to deriving insurance premium and optimal risk exchange, we can completely distinguish tastes from beliefs and risk from ambiguity. In the literature, the search for models of decision-making under ambiguity has been evolving toward the ultimate separation between attitudes for uncertainty and beliefs, and between risk and ambiguity. From this point of view, the paper applying the EUUP in the insurance area is considered to be meaningful.

Contrary to the economic premium principle under the max-min EU by [5], our principle is given unanimously. Furthermore, compared with the economic premium principle under the dual theory of the smooth ambiguity model by [9], our principle can be calculated more easily and explicitly as shown in the numerical examples in the paper. Therefore, it is more appropriate for empirical studies and experiments.

Assuming each agent have an exponential utility function, we also show the optimal insurance or risk exchange decomposes into *individual risk*, *market risk*, and *difference of attitude toward ambiguity*.

Finally, we conduct some comparative statics on insurance premium and risk exchange numerically and examine the influence of attitude for ambiguity aversion on the insurance premium and insurance demand.

First we show the state price density (SPD) or pricing kernel under EUUP is not monotonically deceasing. This is in contrast to the SPD under the Bühlmann model, which monotonically decreases with respect to the state. According to the emprical studeis, the actual SPD takes the bumped shape ([22] [23] [24]). This phenomena is known as "Pricing kernel puzzle"[15]. Hence our model might give a one of explanation of this puzzle.

Next, unlike the result of existing research, we show that each agent's optimal demand for insurance is not always comonotonic even if all the agents in the economy are ambiguity averse or ambiguity loving. That is, even if all the agents in the economy are ambiguity averse or ambiguity loving, insurance trade or risk exchange occurs depending on the difference of attitude fowards ambiguity. For example, [26] and [27] have stated ambiguity seeking is found for losses and un-

[15]The recent survey and discussion on the pricing kernel puzzle is given by [25].

likely events in their experiment. Hence, it is meaningful to consider the economic implications of non-comonotonic risk sharing as treated in the paper from both theoretical and empirical viewpoints.

In the comparative static analysis, we only considered the influence of attitude toward ambiguity, but we also need to consider the influence of the magnitude of ambiguity. This remains for future studies. Furthermore, we need to specify the functional form of the probability outlook function to get more realtic implications. However, we do not know what is a real one to the best of our knowledge. To solve this problem, we must wait for the results of more empirical studies.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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