College Algebra

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Contents

0 Introduction

0.1 About Learning

This textbook is designed to provide you with a basic reference for the topics within. That said, it cannot learn for you, nor can your instructor; ultimately, the responsibility for learning the material lies with you. Before beginning the mathematics, I would like to tell you a little about what research tells us are the best strategies for learning. Here are some of the principles you should adhere to for the greatest success:

- **It's better to recall than to review.** It has been found that re-reading information and examples does little to promote learning. Probably the single most effective activity for learning is attempting to recall information and procedures yourself, rather than reading them or watching someone else do them. The process of trying to recall things you have seen is called *retrieval*.

- **Spaced practice is better than massed practice.** Practicing the same thing over and over (called *massed practice*) is effective for learning very quickly, but it also leads to rapid forgetting as well. It is best to space out, over a period of days and even weeks, your practice of one kind of problem. Doing so will lead to a bit of forgetting that promotes retrieval practice, resulting in more lasting learning. And it has been determined that your brain makes many of its new connections while you sleep!

- **Interleave while spacing.** *Interleaving* refers to mixing up your practice so that you're attempting to recall a variety of information or procedures. Interleaving naturally supports spaced practice.

- **Attempt problems that you have not been shown how to solve.** It is beneficial to attempt things you don't know how to do *if you attempt long enough to struggle a bit*. You will then be more receptive to the correct method of solution when it is presented, or you may discover it yourself!

- **Difficult is better.** You will not strengthen the connections in your brain by going over things that are easy for you. Although your brain is not a muscle, it benefits from being "worked" in a challenging way, just like your body.

- **Connect with what you already know, and try to see the "big picture."** It is rare that you will encounter an idea or a method that is completely unrelated to anything you have already learned. New things are learned better when you see similarities and differences between them and what you already know. Attempting to "see how the pieces fit together" can help strengthen what you learn.

- **Quiz yourself to find out what you *really* do (and don't) know.** Understanding examples done in the book, in class, or on videos can lead to the illusion of knowing a concept or procedure when you really don't. Testing yourself frequently by attempting a variety of exercises without referring to examples is a more accurate indication of the state of your knowledge and understanding. This also provides the added benefit of interleaved retrieval practice.

- **Seek and utilize immediate feedback.** The answers to all of the exercises in the book are in the back. Upon completing any exercise, check your answer right away and correct any misunderstandings you might have. Many of our in-class activities will have answers provided, in one way or another, shortly after doing them.

Through the internet, we now have immediate access to huge amounts of information, some of it good, some of it not! I have attempted to either find or make *quality* videos over many of the concepts and procedures in this book. I have put links to these at my web page,

http://math.oit.edu/~watermang/

(You can also find the web page by doing a search for *waterman oit*.) I would encourage you to view at least a few of them to see whether or not they could be useful for your learning. In keeping with the above suggestions, one good way to use the videos is to view long enough to see the problem to be solved, then pause the video and try solving it yourself. After solving or getting stuck (while trying hard, see the fourth bullet above), forward ahead to see the solution and/or watch how the problem is solved.

It is somewhat inevitable that there will be some errors in this text that I have not caught. As soon as errors are brought to my attention, I will update the online version of the text to reflect those changes. If you are using a hard copy (paper) version of the text, you can look online if you suspect an error. If it appears that there is an uncorrected error, please send me an e-mail at *gregg.waterman@oit.edu* indicating where to find the error.

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May 2015

0.2 About This Course

This course is an in-depth introduction to functions. Our study will be guided by some goals, and related *essential questions* that are designed to get at the essence of the idea of a function.

Course Goals

The overall goal of this course is to understand the concept of a function. This is a multi-faceted task, to be accomplished by developing several more specific understandings related to functions:

- For any type of function studied in the course students will understand, from both analytical (equation) and graphical points of view,
  - the specific behavior of the function as a relationship between two variable quantities (output from input, input from output, domain and range, intercepts, locations and values of minima/maxima)
  - the general behavior of the function as a relationship between two variable quantities (increasing/decreasing, positive/negative, existence of minima/maxima)
- Students will understand the classification of functions into families (linear, quadratic, polynomial, rational, exponential, logarithmic).
- Students will understand the relationship between analytical and graphical representations of functions, including the effects of parameters in the analytical representation on the appearances of graphs.
- Students will understand the role of functions in modelling "real world phenomena."
- Students will understand the role of notation and organization of written work in both clearly communicating ideas and in understanding concepts.

Essential Questions

- How do we determine specific output and input values of interest from either of an equation or a graph representing a function?
- How do we determine general behaviors of a function from its graph, and what behaviors are we interested in?
- What different types of functions are we interested in? How are the various types alike, and how are they different?
- How are the graphs of functions affected by changes to parameters in their equations?
- How do we use mathematical functions to represent related quantities in science, engineering and finance?
- What are the commonly accepted notations that allow us to communicate mathematics with others? How do we organize work so that others can understand our thought processes?

0.3 An Example

Much of mathematics concerns itself with describing, as precisely as possible, relationships between related variable quantities. Although in reality several quantities are often related, we will stick with relationships between two quantities in this course. In such situations, one quantity usually depends on the other. Here are some examples one might run across:

- The cost of building a fence depends on the length of the fence.
- The number of fish in a lake depends on the length of time after some "starting" time.
- The intensity of an ultrasound signal depends on how far it has traveled through soft tissue.
- The demand for some item depends on the cost of the item.

If quantity y depends on quantity x, we call y the **dependent variable** and x the **independent variable**.

We usually want to obtain an equation that describes such a relationship, or to use such an equation that is already known. An equation relating variable quantities is an example of what we call a **mathematical model**. Models (equations) are usually obtained from either statistical data or physical principles.

Lets take a look at an example of a model and what we might do with it. In doing so we will touch on the major themes of this entire course. Suppose that the Acme Company makes and sells Geegaws. Their weekly profit P (in dollars) is given by the equation

$$P = -8000 + 27x - 0.01x^2,$$

where x is the number of Geegaws sold that week. Note that it will be possible for P to be negative (for example, if they make and sell $x = 0$ Geegaws in a week, their profit is $P = -8000$ dollars for the week); of course negative profit is really loss!

◇ **Example 0.1(a):** If 1280 Geegaws are sold in a week, what profit can the Acme Company expect for the week?

Solution: This is the most basic use for our model, and we obtain the profit by **evaluating the equation** for the value $x = 1280$:

$$P = -8000 + 27(1280) - 0.01(1280)^2 = \$10,176$$

◇ **Example 0.1(b):** Determine the number of Geegaws that must be sold in a week in order to obtain a profit of $7,000.

Solution: In this situation the variable P has the value 7000, and we must use that fact to determine the corresponding value of x. Substituting the value of P in, we obtain

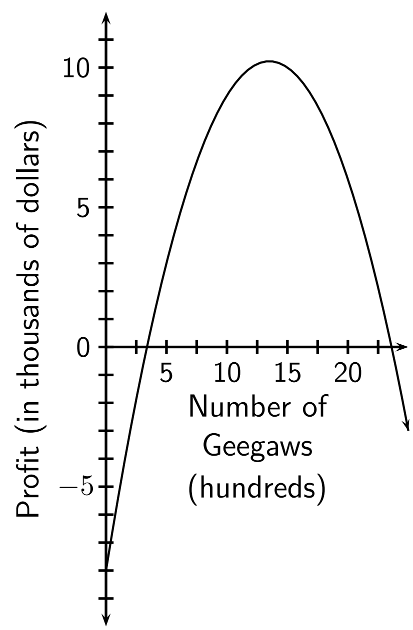
$$7000 = -8000 + 27x - 0.01x^2.$$

Here we can't simply evaluate like we did in the previous example - we must instead **solve the equation**. You probably already know the technique for doing this, but we will review it in the next section. It turns out that the equation has *two* solutions, $x = 782$ Geegaws or $x = 1918$ Geegaws. Producing and selling either of those numbers of Geegaws will result in a weekly profit of $7,000. It may not be clear why, but we'll save for later the discussion of why that is the case.

The two processes you've just seen carried out will be repeated over and over in this course. Given a model for the relationship between two variables, we often want to find the value of one of the variables, given the value of the other. In some cases we will be able to simply evaluate an equation, and in other cases we'll have to solve it. The techniques of solving equations vary somewhat, depending on the type of equation we have. Determining the correct technique for solving an equation, and carrying that technique out, will be one of our areas of focus.

Evaluating or solving are often useful, but they fail to give us an overall understanding of the nature of a relationship between two variables. When we wish to "see the forest rather than individual trees" we will create and evaluate a **graph of the equation**. You are undoubtedly familiar with this idea, but let's take a look at it for our equation. Below and to the right is a graph of the equation. Note that the independent variable is (always) on the horizontal axis and the dependent variable is (always) on the vertical axis. From the graph we can make a number of observations:

- The profit can actually be negative if few Geegaws are made and sold, and it is also negative when too many are made. When few Geegaws are produced and sold, the amount of money brought in (called the **revenue**) is not high enough to pay the fixed costs of maintaining a facility and paying employees.



- Initially the profit increases as more Geegaws are made and sold, but eventually the profit starts decreasing as more Geegaws are produced. Making and selling more Geegaws results in greater profit until too many are produced. At that point the supply exceeds the demand and the excess Geegaws cannot be sold. Eventually the profit decreases to the point that it is negative, because the cost of producing unsold Geegaws exceeds the revenue generated by the ones that are sold.

- Where the profit quits increasing and starts decreasing, the maximum profit is obtained. We can see that this occurs when the number of Geegaws produced and sold is about 1300 or 1400. We'll see later how to determine the exact number of Geegaws that need to be produced and sold to maximize the profit. (This can be done precisely from the equation itself, rather than relying on the graph to obtain an approximate value.)

One other thing we might wish to do is determine the number of Geegaws that need to be produced in order to generate some minimum weekly profit, like at least $5000. In this case we want the values of x for which the profit is greater than or equal to $5000:

$$-8000 + 27x - 0.01x^2 \geq 5000$$

From the graph we can estimate that Acme needs to make and sell between about 700 and 2200 Geegaws to obtain a profit of at least $5000, but we will see later that we can use the equation to get a more precise answer. For now, let's get on with the basics of solving equations and graphing relationships.

1 Equations in One and Two Unknowns

Outcome/Performance Criteria:

1. Solve equations in one unknown, and apply equations to solve problems. Graph solution sets of equations in two unknowns.

   (a) Solve a linear, quadratic, polynomial and absolute value equation.

   (b) Solve an equation containing rational or radical expressions.

   (c) Solve an applied problem by writing and solving an equation modeling the given situation.

   (d) Use a formula to solve a problem.

   (e) Solve a formula for a given unknown.

   (f) Find solution pairs for an equation in two unknowns.

   (g) Graph the solution set of an equation in two unknowns without using a graphing calculator.

   (h) Identify x- and y-intercepts of an equation in two unknowns from its graph; find x- and y-intercepts of an equation in two unknowns algebraically.

   (i) Graph a line; determine the equation of a line.

   (j) Solve a system of two linear equations in two unknowns by the addition method; solve a system of two linear equations in two unknowns by the substitution method.

   (k) Solve a problem using a system of two linear equations in two unknowns.

   (l) Graph $y = ax^2 + bx + c$ by finding and plotting the vertex, y-intercept, and several other points.

   (m) Identify or create the graphs of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

   (n) Graph variations of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

1.1 Solving Equations in One Unknown, Part I

Performance Criterion:

1. (a) Solve a linear, quadratic, polynomial or absolute value equation.

The key idea in solving an equation is the following:

We create a sequence of new equations that are equivalent to the original equation, until we get one that is simple enough that we can see what the solution(s) is (are).

There is no single procedure that can be used to solve every equation, but there are some general guidelines to keep in mind. The guidelines will be illustrated in this section and the next one, and summarized at the end of that section.

Linear Equations

Linear equations are ones in which the unknown is *NOT* to a power, under a root, or in the bottom of a fraction. Here are some examples of linear equations:

$$5 - 2t = 1 \qquad 20 = 8 - 5(2x - 3) + 4x \qquad \tfrac{1}{2}x + \tfrac{1}{4} = \tfrac{1}{3}x + \tfrac{5}{4}$$

Here are some examples of equations that *ARE NOT* linear:

$$3x^3 + 36x = 21x^2 \qquad \frac{x}{x-4} + \frac{2x}{x^2 - 7x + 12} = \frac{15}{x-3}$$

$$t + 5 = \sqrt{t+7} \qquad (x-2)(x+1) = 5$$

Let's look a bit more at the two equations

$$(x-2)(x+1) = 5 \qquad \text{and} \qquad 20 = 8 - 5(2x-3) + 4x$$

The equation on the left might appear to be linear, since the two places we see an x it is to the first power and not under a root or in the bottom of a fraction. However, if we distribute to eliminate parentheses, the two equations become

$$x^2 - x - 2 = 5 \qquad \text{and} \qquad 20 = 8 - 10x + 15 + 4x$$

and we can now see that the first of these two is not linear.

To solve linear equations we usually follow these steps:

1) Eliminate parentheses by distributing; eliminate fractions by multiplying both sides by the least common denominator of all fractions.

2) Combine like terms on each side of the equation.

3) Add or subtract terms to or from both sides of the equation to get a term with the unknown on one side, and just a number on the other.

4) Divide by the coefficient of the unknown (the number multiplying it) on both sides to finish.

Let's demonstrate the above steps with some examples.

◇ **Example 1.1(a):** Solve $5 - 2t = -1$. Another Example

Solution: Here is how we would probably solve this equation:

$5 - 2t = -1$	the original equation
$-2t = -6$	subtract 5 from both sides
$t = 3$	divide both sides by -2

To avoid having negatives at the second step we could instead add $2t$ and 1 to both sides:

$5 - 2t = -1$	the original equation
$6 = 2t$	add $2t$ and 1 to both sides
$3 = t$	divide both sides by 2

◇ **Example 1.1(b):** Solve $20 = 8 - 5(2x - 3) + 4x$. Another Example

Solution:

$20 = 8 - 5(2x - 3) + 4x$	the original equation
$20 = 8 - 10x + 15 + 4x$	distribute to eliminate parentheses
$20 = -6x + 23$	combine like terms
$6x = 3$	add $6x$ and subtract 20 to/from both sides
$x = \frac{1}{2}$	divide both sides by 6 and reduce

◇ **Example 1.1(c):** Solve $\frac{1}{2}x + \frac{1}{4} = \frac{1}{3}x + \frac{5}{4}$. Another Example

Solution:

$\frac{1}{2}x + \frac{1}{4} = \frac{1}{3}x + \frac{5}{4}$	the original equation
$\frac{12}{1}\left(\frac{1}{2}x + \frac{1}{4}\right) = \frac{12}{1}\left(\frac{1}{3}x + \frac{5}{4}\right)$	multiply both sides by 12 in the form $\frac{12}{1}$
$\frac{12}{1}\left(\frac{1}{2}x\right) + \frac{12}{1}\left(\frac{1}{4}\right) = \frac{12}{1}\left(\frac{1}{3}x\right) + \frac{12}{1}\left(\frac{5}{4}\right)$	distribute the $\frac{12}{1}$ to each term
$6x + 3 = 4x + 15$	multiply - fractions are gone!
$2x = 12$	get x terms on one side, numbers on other
$x = 6$	divide both sides by 2

Quadratic and Polynomial Equations

A **polynomial expression** consists of some whole number powers of an unknown, each multiplied by a number (called a **coefficient**), and all added together. Some examples would be

$$x^4 - 7x^3 + 5x^2 + \tfrac{2}{3}x - 1 \qquad \text{and} \qquad 3.72x^2 + 14.7x$$

Polynomial equations are equations in which each side of the equation is a polynomial expression (which can be just a number). Here are some examples of polynomial equations:

$$2x^2 + x = 3 \qquad 3x^3 + 36x = 21x^2 \qquad 9x^4 = 12x^3 \qquad x^2 - 10x + 13 = 0$$

The highest power of x is called the **degree** of the polynomial equation. The highest power of the unknown *CAN* be first power, so linear equations are in fact polynomial equations. We generally don't want to think of them as such, since our method for solving them is a bit different than what we do to solve polynomial equations of degree two and higher. Polynomial equations of degree two are the most commonly encountered polynomial equations, and they have their own name: **quadratic** equations. The general procedure for solving polynomial equations is this:

1) Add or subtract all terms to get them on one side of the equation and zero on the other side.

2) Factor the non-zero side of the resulting equation.

3) Determine the values of the unknown that will make each factor zero; those values are the solutions to the equation.

Here are some examples of this process in action:

◇ **Example 1.1(d):** Solve $2x^2 + x = 3$. Another Example

Solution:

$2x^2 + x = 3$		the original equation
$2x^2 + x - 3 = 0$		get all terms on one side and zero on the other
$(2x+3)(x-1) = 0$		factor
$2x + 3 = 0$	or $x - 1 = 0$	set each factor equal to zero to get linear equations
$x = -\frac{3}{2}$	or $x = 1$	solve linear equations

It should have been clear after factoring that one of the solutions is 1, and there is really no need to show the equation $x - 1 = 0$.

Note that we should have been able to see at the third step above that if x was one we would have $(something)(0)$ on the left, which of course is zero regardless of what the "something" is. This could be called "solving by inspection" at that point. As another example, if we had $(x+5)(x-2) = 0$, we could see that the left side would be zero if x is either of -5 or 2, so those are the solutions to the equation. This method will be used in the following example:

◇ **Example 1.1(e):** Solve $3x^3 + 36x = 21x^2$. Another Example

Solution:

$3x^3 + 36x = 21x^2$	the original equation
$3x^3 - 21x^2 + 36x = 0$	get all terms on one side and zero on the other
$3x(x^2 - 7x + 12) = 0$	factor out the common factor of $3x$
$3x(x-3)(x-4) = 0$	factor $x^2 - 7x + 12$
$x = 0,\ 3,\ 4$	obtain solutions by inspection

We should make one observation at this point. In each of the examples we have seen so far, the number of solutions to each equation is the degree of the equation. This is not always the case, but it is "close." The facts are these:

> The number of solutions to a polynomial equation is the degree of the equation or less. A linear equation has one solution or no solution.

Consider now the equation $x^2 = 25$. From what we just read we would expect this equation to perhaps have two solutions. We might think that we can just square root both sides of the equation to get the solution $x = 5$, but then we have one less solution than we might expect. Moreover, we can see that $(-5)^2 = 25$, so -5 is also a solution, giving us the two solutions that we expect. The moral of this story is that we *can* take the square root of both sides of an equation if we then consider both the positive and negative of one side, usually the side with only a number. Here are two examples:

◇ **Example 1.1(f):** Solve $9x^2 = 16$.

Solution:

$9x^2 = 16$	the original equation
$x^2 = \frac{16}{9}$	divide both sides by 9 to get x^2 alone on one side
$x = \pm\sqrt{\frac{16}{9}}$	take the square root of both sides, including both the positive and negative square root on the right side
$x = \pm\frac{4}{3}$	evaluate the square root

◇ **Example 1.1(g):** Solve $3x^2 = 60$.

Solution:

$3x^2 = 60$	the original equation
$x^2 = 20$	divide both sides by 3 to get x^2 alone on one side
$x = \pm\sqrt{20}$	take the square root of both sides, including both the positive and negative square root on the right side
$x = \pm 2\sqrt{5}$ or $x = \pm 4.47$	simplify the square root and give the decimal form

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Of course we should be able to solve at least the equation from Example 1.1(f) by getting zero on one side and factoring, our standard approach to solving polynomial equations:

◇ **Example 1.1(h):** Solve $9x^2 = 16$.

Solution:

$9x^2$	$=$	16	the original equation
$9x^2 - 16$	$=$	0	get all terms on one side and zero on the other
$(3x+4)(3x-4)$	$=$	0	factor the difference of squares
$3x+4=0$	or	$3x-4=0$	set each factor equal to zero to get linear equations
$x=-\frac{4}{3}$	or	$x=\frac{4}{3}$	solve linear equations

Sometimes quadratic equations can't be solved by factoring. In those cases we use the quadratic formula.

Quadratic Formula

$$\text{If } ax^2+bx+c=0, \text{ then } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

◇ **Example 1.1(i):** Use the quadratic formula to solve $x^2 - 10x + 13 = 0$. Give your answers as two separate numbers, in simplified square root form and in decimal form, rounded to the nearest hundredth. Another Example

Solution: We first determine the values of a, b and c: $a = 1$, $b = -10$, $c = 13$. We then substitute them into the quadratic formula and evaluate carefully as follows:

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(13)}}{2(1)}=\frac{10\pm\sqrt{100-52}}{2}=\frac{10\pm\sqrt{48}}{2}$$

At this point we simplify $\sqrt{48}$ to $4\sqrt{3}$, Then either factor a common factor of two out of the top and bottom and reduce, or split the fraction in two and reduce each part:

$$x=\frac{10\pm4\sqrt{3}}{2}=\frac{2(5\pm2\sqrt{3})}{2}=5\pm2\sqrt{3}$$

or

$$x=\frac{10\pm4\sqrt{3}}{2}=\frac{10}{2}\pm\frac{4\sqrt{3}}{2}=5\pm2\sqrt{3}$$

Decimal solutions are best computed from the form $\frac{10\pm\sqrt{48}}{2}$, giving $x = 1.54,\ 8.46$.

Let's now look at an application of quadratic equations and the quadratic formula. For most of the remainder of this course we will be seeing the following situation: We will have, or create, an equation containing two related unknown quantities. We will be given a value for one of the quantities, and we will need to evaluate an expression or solve an equation to find the corresponding value of the other unknown quantity.

Projectile Motion

Suppose that an object is projected straight upward from a height of h_0 feet, with an initial velocity of v_0 feet per second. Its height (in feet) at any time t seconds after it is projected (until it hits the ground) is

$$h = -16t^2 + v_0 t + h_0$$

There is some terminology associated with the above formula that many of you will hear again. The quantities h_0 and v_0 are called **parameters**, which means that they can change from situation to situation, but for a given situation they remain constant. (The -16 is also a parameter, but it would only change if we went to a different planet!) The quantities h and t are **variables**; they vary as the situation we are studying "unfolds." They are related to each other in the way described mathematically by the given equation. Let's look at a couple examples using this formula.

◇ **Example 1.1(j):** A ball is thrown upward from a height of 5 feet, with an initial velocity of 60 feet per second. How high is the ball after 1.5 seconds?

Solution: Here the parameters h_0 and v_0 have values 5 and 60, respectively, so the equation becomes $h = -16t^2 + 60t + 5$. Now we substitute 1.5 in for t and compute the corresponding value for h:

$$h = -16(1.5)^2 + 60(1.5) + 5 = 59$$

The height of the ball after 1.5 seconds is 59 feet.

◇ **Example 1.1(k):** For the same ball, at what time or times is the ball 50 feet above the ground? Round the answers to the nearest hundredth of a second.

Solution: In this case we can see that we are given h and asked to find t, so we must solve the equation $50 = -16t^2 + 60t + 5$. Subtracting 50 from both sides we get $-16t^2 + 60t - 45 = 0$. We can then solve with the quadratic formula, to get $t = 1.04$ seconds and $t = 2.71$ seconds. There are two times because the rock is at a height of 50 feet on the way up, and again on the way down!

Absolute Value Equations

Most of you are probably familiar with the concept of absolute value, but let's do a brief review. First, when we see something like $-a$, this means "the opposite of a." If a is positive, then $-a$ is negative; however, if a is negative, $-a$ is then positive. The upshot of all this is that something like $-x$ does not necessarily represent a negative number.

The **absolute value** of a number x is just x itself if x is zero or positive. If x is negative, then the absolute value of x is $-x$, a positive number. We use the notation $|x|$ to represent the absolute value of x. All the words just given can be summarized concisely using mathematical notation:

Absolute Value

Let $|x|$ represent the absolute value of a number x. Then $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

◇ **Example 1.1(l):** Find $|4.2|$ and $|-37|$.

Solution: $|4.2| = 4.2$ and $|-37| = -(-37) = 37$

◇ **Example 1.1(m):** Find all values of x for which $|x| = 5$.

Solution: $x = \pm 5$, since both $|5| = 5$ and $|-5| = 5$.

The only absolute value equations that we will solve are *linear* absolute value equations, which basically means they are linear equations with some absolute values in them. Suppose that we have an equation of the form $|\,stuff\,| = 7$. Based on the last example, that means that *stuff* has to be either 7 or -7. That is the key to how we solve absolute value equations.

◇ **Example 1.1(n):** Solve $|3x-5| = 7$. Another Example

Solution:

$\lvert 3x-5\rvert$	$=$	7	the original equation
$3x-5=-7$	or	$3x-5=7$	write as two separate equations
$3x=-2$	or	$3x=12$	add 5 to both sides of each equation
$x=-\frac{2}{3}$	or	$x=4$	divide both sides of each equation by 3

Section 1.1 Exercises

To Solutions

1. Solve each of the following linear equations.

(a) $4x+7=5$

(b) $2x-21=-4x+39$

(c) $\frac{9}{5}x-1=2x$

(d) $7(x-2)=x+2(x+3)$

(e) $8x=10-3x$

(f) $\frac{x}{4}+\frac{1}{2}=1-\frac{x}{8}$

(g) $\frac{2}{3}x+\frac{1}{2}=\frac{3}{2}x-\frac{1}{6}$

(h) $3(x-5)-2(x+7)=3x+1$

2. Solve each polynomial equation.

(a) $x^2 - 13x + 12 = 0$ (b) $x^3 + 3x^2 + 2x = 0$ (c) $5x^2 + x = 0$

(d) $\frac{2}{3}x^2 + \frac{7}{3}x = 5$ (e) $3a^2 + 24a = -45$ (f) $8x^2 = 16x$

(g) $3x^2 - 48 = 0$ (h) $\frac{1}{15}x^2 = \frac{1}{6}x + \frac{1}{10}$ (i) $x^3 = 9x^2 + 22x$

3. For each of the following, use the quadratic formula to solve the given equation. Give your answers in both simplified square root form and decimal form rounded to the nearest tenth.

(a) $x^2 - 2x - 4 = 0$ (b) $x^2 = 2x + 17$

(c) $2x^2 = 2x + 1$ (d) $25x^2 + 10x = 6$

4. Solve each of the following absolute value equations.

(a) $|3x + 4| = 2$ (b) $|7x - 1| = 3$ (c) $|x^2 - 13| = 3$

5. Solve the equation $2x^2 + x = 3$ from Example 1.1(d) using the quadratic formula. This shows that even if a quadratic equation can solved by factoring, the quadratic formula will give the solutions as well.

6. A baseball is hit upward from a height of four feet and with an initial velocity of 96 feet per second.

(a) When is it at a height of 84 feet? Note that 16 goes into 80 and 96 evenly.

(b) When is it at a height of 148 feet? Note that 16 goes into 96 and 144 evenly.

(c) When is it at a height of 52 feet? Use your calculator and the quadratic formula, and round your answer(s) to the nearest hundredth of a second. (The hundredth's place is two places past the decimal.)

(d) When is the ball at a height of 200 feet, to the nearest hundredth of a second?

(e) When does the ball hit the ground, to the nearest hundredth of a second?

7. A company that manufactures ink cartridges for printers knows that the number x of cartridges that it can sell each week is related to the price per cartridge p by the equation $x = 1200 - 100p$. The weekly revenue (money they bring in) is the price times the number of cartridges: $R = p(1200 - 100p)$. What price should they set for the cartridges if they want the weekly revenue to be $3200?

8. (a) Use the strategy of getting all terms on one side of the equation and factoring to solve $9x^2 = 12x$.

   (b) It would seem reasonable to try to solve the same equation by first dividing both sides by $3x$. What happens if you do that? What went wrong?

9. Students often think that when they are solving a polynomial equation with zero on one side and they can factor out a common factor, zero is one of the solutions. Give an example where this *IS* the case, and another where it *IS NOT*.

10. Solve the three equations $4x^2 = 25x$, $4x^2 = 25$, and $4x^3 = 25x^2$.

11. Consider the equation $(x-1)(x^2+5x+6) = 0$.

   (a) If you were to multiply the left side out (don't do it!) you would get a polynomial. What would its degree be? How many solutions do you expect to find for this equation?

   (b) The left side of this equation contains the product of two factors A and B. What are they?

   (c) What value or values cause A to be zero?

   (d) What value or values cause B to be zero? (You'll have to work a little harder here!)

   (e) What are the solutions to the original equation? Does this agree with what you got in (a)?

12. Solve $(x-2)(x^2-8x+7) = 0$.

13. Solve each of the following equations. *Note that each is already factored!*

   (a) $(x+1)(x-3)(x-2) = 0$ (b) $(x-3)(x-4)(x+1)(x-1) = 0$

   (c) $(x+2)^2(x-5) = 0$

14. Solve $|2x-3| = -7$. (Trick question! Explain.)

15. When solving an equation, if we arrive at $(x+3)(x-1) = 0$ we conclude that $x+3 = 0$ or $x-1 = 0$, so $x = -3, 1$.

   (a) A student solving a *different* equation correctly arrives at $(x+3)(x-1) = 12$. They then write $x+3 = 12$ and $x-1 = 12$ and, from those, reach the conclusion that $x = 9, 13$. Convince them that they are wrong, *without solving the equation*.

   (b) Solve the equation $(x+3)(x-1) = 12$, and verify that your answers are correct.

16. Use the quadratic formula to solve each of the following:

   (a) The equation from Example 1.1(g) - check your answer with the example.

(b) The equation $3x^2 = 5x$ - check your answer by also solving by factoring.

17. The yellow box after Example 1.1(e) tells us that a quadratic equation can have two, one or zero solutions. You have seen a number of quadratic equations with two solutions - here you will see equations with one and zero solutions.

    (a) Solve the equation $x^2 + 9 = 6x$. Check your answer by substituting it into the equation.

    (b) Explain why $x^2 + 9 = 0$ doesn't have a solution.

18. Solve the equation $-\frac{1}{4}x^4 + 2x^2 - 4 = 0$ by the following method:

    (a) Factor $-\frac{1}{4}$ out of the left side. (As usual, though, *don't get rid of it!*) Multiply it back in mentally to make sure you factored correctly.

    (b) Factor the remaining part in pretty much the same way as you would if it were $x^2 + bx + c$, except the powers will be a little different. "FOIL" what you get back together to make sure it is correct.

    (c) Now you should be able to do more factoring to finish solving.

19. Solve each equation. For those that require the quadratic formula, give your answers in both exact and decimal form, rounded to the nearest tenth.

    (a) $4x - 3(5x + 2) = 4(x + 3)$

    (b) $x^2 - 19 = 6x$

    (c) $\dfrac{2}{3}x - 1 = \dfrac{3}{4}x + \dfrac{1}{2}$

    (d) $3|x - 1| + 2 = 4$

    (e) $9x^2 - 12x - 1 = 0$

    (f) $x^3 - 7x^2 + 10x = 0$

    (g) $|x + 2| - 5 = 0$

    (h) $11x + 9 = 3x + 11$

    (i) $6x^2 + 13x = 5$

    (j) $x^2 + 8x + 13 = 0$

    (k) $\dfrac{2}{15}x^2 + \dfrac{1}{3}x = \dfrac{1}{5}$

    (l) $21 + 4x = x^2$

1.2 Solving Equations in One Unknown, Part II

Performance Criterion:

1. (b) Solve an equation containing rational or radical expressions.

Equations With Rational Expressions

A **rational expression** is a fraction whose numerator and denominator are both polynomial expressions. Some examples are

$$\frac{24}{x+5} \qquad \text{and} \qquad \frac{x^2-25}{x^2-7x+12}.$$

The general strategy for solving an equation with rational expressions in it is this:

1) Factor the denominators of all rational expressions.
2) Multiply both sides of the equation by *just enough* of the factors of the denominators to eliminate all fractions.
3) The result after (2) will be a linear or polynomial equation. Solve it.
4) Check to see whether any of the solutions that you obtained causes a denominator to be zero in the original equation; if it does, eliminate it.

◇ **Example 1.2(a):** Solve $7 = \frac{3x-2}{2x-5}$.

Solution: If we multiply both sides by $2x-5$ the denominator on the right will be eliminated. At that point we will have a linear equation, which we already know how to solve.

$7 = \frac{3x-2}{2x-5}$		the original equation
$\frac{2x-5}{1}(7) = \frac{2x-5}{1}\left(\frac{3x-2}{2x-5}\right)$		multiply both sides by $\frac{2x-5}{1}$
$14x-35 = 3x-2$		distribute the 7 on the left side, cancel the $2x-5$ on the right
$11x = 33$		get x terms on the left, number terms on the right
$x = 3$		finish solving

At this point we check to see that $x=3$ does not cause the denominator $2x-5$ of the original equation to be zero. Since it doesn't, the solution is $x=3$ (providing there are no errors in the computations).

◇ **Example 1.2(b):** Solve $\dfrac{1}{x+1}+\dfrac{11}{x^2-x-2}=\dfrac{3}{x-2}$.

Solution: Here we first factor the denominator x^2-x-2. At that point we can see that if we multiply both sides by $(x+1)(x-2)$ we will eliminate all fractions.

$$\begin{aligned}
\frac{1}{x+1}+\frac{11}{x^2-x-2} &= \frac{3}{x-2}\\
\frac{1}{x+1}+\frac{11}{(x+1)(x-2)} &= \frac{3}{x-2}\\
\frac{(x+1)(x-2)}{1}\left(\frac{1}{x+1}+\frac{11}{(x+1)(x-2)}\right) &= \frac{(x+1)(x-2)}{1}\left(\frac{3}{x-2}\right)\\
(x-2)+11 &= 3(x+1)\\
x+9 &= 3x+3\\
6 &= 2x\\
3 &= x
\end{aligned}$$

At this point we check to see that $x=3$ does not cause any of the denominators of the original equation to be zero. Since it doesn't, the solution is $x=3$ (providing there are no errors in the computations).

In the next example we will see a situation where two solutions are obtained and one of them is not valid.

◇ **Example 1.2(c):** Solve $\dfrac{a+4}{a^2+5a}=\dfrac{-2}{a^2-25}$.

Solution: For this example we begin by factoring the denominators of the original equation $\dfrac{a+4}{a^2+5a}=\dfrac{-2}{a^2-25}$ to get $\dfrac{a+4}{a(a+5)}=\dfrac{-2}{(a+5)(a-5)}$. Now we can see that if we multiply both sides by $a(a+5)(a-5)$ we will eliminate the denominators:

$$\begin{aligned}
\frac{a(a+5)(a-5)}{1}\left(\frac{a+4}{a(a+5)}\right) &= \frac{a(a+5)(a-5)}{1}\left(\frac{-2}{(a+5)(a-5)}\right)\\
(a-5)(a+4) &= -2a
\end{aligned}$$

Finally we multiply out the left side and solve the resulting quadratic equation:

$$\begin{aligned}
a^2-a-20 &= -2a\\
a^2+a-20 &= 0\\
(a+5)(a-4) &= 0
\end{aligned}$$

Here we can see that $a=-5$ and $a=4$ are solutions to the last equation. Looking back at the original equation, however, shows that a cannot be -5 because that value causes the denominators of both sides to be zero. Thus the only solution is $a=4$.

Equations With Radical Expressions

A **radical expression** is simply a root with an unknown under it. We will occasionally wish to solve equations containing such expressions. The procedure for solving an equation with just one radical expression in it goes like this:

1) Add or subtract on each side to get the root alone on one side.

2) Square both sides of the equation. Note that *you must square the entire side*, not the individual terms.

3) The result of the previous step will be a linear or quadratic equation. Solve it.

4) *You must check all solutions in the original equation* for this type of equation. The process of squaring both sides of the equation sometimes introduces solutions that are not valid.

◇ **Example 1.2(d):** Solve $\sqrt{3x+7} = 4$.

Solution: Here we begin by simply squaring both sides of the original equation, then just solve the resulting equation:

$$\begin{aligned}
\sqrt{3x+7} &= 4 && \text{the original equation}\\
(\sqrt{3x+7})^2 &= 4^2 && \text{square both sides}\\
3x+7 &= 16 && \text{squaring eliminates the square root}\\
3x &= 9 && \text{subtract } 7 \text{ from both sides}\\
x &= 3 && \text{divide both sides by } 3
\end{aligned}$$

Checking, we see that $\sqrt{3(3)+7} = \sqrt{16} = 4$, so $x = 3$ is in fact the solution to the original equation.

◇ **Example 1.2(e):** Solve $t = \sqrt{t+7} - 5$.　　Another Example - stop at 5:15

Solution: In this case we need to add 5 to both sides of the original equation to get the square root alone on one side, *THEN* square both sides:

$$\begin{aligned}
t &= \sqrt{t+7} - 5 && \text{the original equation}\\
t+5 &= \sqrt{t+7} && \text{add } 5 \text{ to both sides}\\
(t+5)^2 &= (\sqrt{t+7})^2 && \text{square both sides}\\
(t+5)(t+5) &= t+7 && \text{squaring the left side means}\\
t^2+10t+25 &= t+7 && \text{multiplying by itself}\\
t^2+9t+18 &= 0 && \text{get all terms on one side}\\
(t+3)(t+6) &= 0 && \text{factor}\\
t = -3 & \qquad t = -6 && \text{finish solving}
\end{aligned}$$

Let's check both of these solutions in the original equation, using $\stackrel{?}{=}$ in place of $=$ until we are certain whether the statements are true:

$$\begin{aligned} -3 &\stackrel{?}{=} \sqrt{(-3)+7}-5 \\ -3 &\stackrel{?}{=} \sqrt{4}-5 \\ -3 &= -3 \end{aligned} \qquad \begin{aligned} -6 &\stackrel{?}{=} \sqrt{(-6)+7}-5 \\ -6 &\stackrel{?}{=} \sqrt{1}-5 \\ -6 &\neq -4 \end{aligned}$$

We can see that $t=-3$ is a solution but $t=-6$ is not.

What happens when we square both sides of an equation is that any solutions to the original equation are still solutions to the new equation, but the new equation will often have additional solutions. A simple example is the equation $x=5$, whose only solution is obviously $x=5$. When we square both sides we get the equation $x^2=25$. Note that $x=5$ is still a solution to this new equation, but $x=-5$ is as well. However $x=-5$ is not a solution to the original equation $x=5$.

General Strategies For Solving Equations

Here is the summary of equation-solving techniques that was promised earlier:

Techniques For Solving Equations

- If there are grouping symbols in the equation, it is *usually* beneficial to distribute in order to eliminate them.
- If the equation contains fractions, multiply both sides by the common denominator to eliminate all fractions.
- If the equation contains roots, get one root alone on one side and apply the corresponding power to both sides of the equation. Repeat if necessary.
- If the equation contains only the unknown, not to any powers, under roots or in denominators of fractions, it is a linear equation. Get all terms with x on one side, other terms on the other side, simplify and solve.
- If the equation contains various powers of x, get all terms on one side of the equation and zero on the other side and factor. Finish solving by inspection or by setting factors equal to zero and solving the resulting equations.
- If the equation is quadratic and you can't factor it, try the quadratic formula.

If the original equation contained x under roots or in the denominators of fractions, one generally needs to check all solutions, since invalid solutions can be created by those methods. If there are solutions, you will be guaranteed to find them, but you might also come up with solutions that are not really solutions.

Section 1.2 Exercises

To Solutions

1. Solve each equation. Be sure to check all solutions to see if they are valid.

   (a) $\dfrac{x+4}{2x} + \dfrac{x+20}{3x} = 3$

   (b) $\dfrac{x+1}{x-3} = \dfrac{x+2}{x+5}$

   (c) $1 - \dfrac{4}{x+7} = \dfrac{5}{x+7}$

   (d) $\dfrac{4x}{x+2} = 4 - \dfrac{2}{x-1}$

   (e) $\dfrac{1}{x-1} + \dfrac{1}{x+1} = \dfrac{6}{x^2-1}$

   (f) $\dfrac{2x-1}{x^2+2x-8} = \dfrac{1}{x-2} - \dfrac{2}{x+4}$

2. Solve each equation. Be sure to check all solutions to see if they are valid.

   (a) $\sqrt{25x-4} = 4$

   (b) $\sqrt{2x+7} - 6 = -2$

   (c) $\sqrt{3x+13} = x+3$

   (d) $\sqrt{2x+11} - 4 = x$

   (e) $\sqrt{3x+2} + 7 = 5$

   (f) $\sqrt{x-2} = x-2$

3. Solve each equation.

   (a) $\sqrt{x-2} + 3 = 5$

   (b) $\dfrac{x+3}{x-5} = -2$

   (c) $\dfrac{3(x-2)(x+1)}{(x-4)(x+4)} = 0$

   (d) $\sqrt{x+5} - 1 = 0$

   (e) $\dfrac{x^2-1}{x^2-3x-2} = 3$

   (f) $\sqrt{x+2} + 4 = 0$

4. Later in the term we will see equations like

$$30 = \frac{120}{1+200e^{-0.2t}}$$

   that we'll want to solve for t. Using the techniques of this section and Section 1.1, solve the equation for $e^{-0.2t}$.

5. (a) Solve $\dfrac{3x-1}{x^2+2x-15} = 0$.

   (b) Solve $\dfrac{x^2-3x-4}{x^2+2x+1} = 0$

   (c) Note what you got after you cleared the fractions in each of the above. From this we can see that in order to solve $\dfrac{A}{B} = 0$ we need simply to solve ..."

6. Solve each equation. When the quadratic equation is required, give both exact solutions and approximate (decimal) solutions rounded to the nearest tenth.

(a) $2x^2 = x + 10$

(b) $\dfrac{4}{x-3} - \dfrac{3}{x+3} = 1$

(c) $10x^3 = 29x^2 + 21x$

(d) $\dfrac{3x}{4} - \dfrac{5}{12} = \dfrac{5x}{6}$

(e) $|3x + 2| = 10$

(f) $x^2 + 10x + 13 = 0$

(g) $\dfrac{1}{5}x^2 = \dfrac{1}{6}x + \dfrac{1}{30}$

(h) $\sqrt{5x+9} = x - 1$

1.3 Solving Problems with Equations

Performance Criteria:

1. (c) Solve an applied problem by writing and solving an equation modeling the given situation.
   (d) Use a formula to solve a problem.
   (e) Solve a formula for a given unknown.

In this section and the next we will use equations to solve problems. (Specifically, what you have probably called "word problems.") Given such a problem, the object will be to write an equation, or use one that we already know, whose solution is the solution to the problem. (In some cases the solution to the equation will only be part of the solution to the problem.) Of course the most difficult part of this is finding the equation. If you are able to write an equation directly, I would welcome you to do so. However, if you have trouble writing the equation, I would suggest you use the following strategy:

- Get to know the situation by guessing a value for the thing you are supposed to find, then doing the appropriate arithmetic to check your guess. (If you are supposed to find several quantities, just guess the value of one of them.) *Do not spend a lot of time and effort trying to decide on your guess - it will probably not be the correct solution, but we don't need it to be!*
- Let x be the correct value that you are looking for. Do everything to x that you did to your guess in order to check it, but set the result equal to what it is really supposed to be. This will be the equation you are looking for!
- Solve the equation to get the value asked for. If asked for more than one value, find the others as well.

Let's look at some examples to see how this process works.

◇ **Example 1.3(a):** A company is billed $645 for the services of an engineer and a technician for a job. The engineer is billed out at $90 per hour, and the technician at $60 per hour. The technician worked on the project for two hours more than the engineer did. How long did each work?

Solution: Let's suppose that the engineer worked on the job for 5 hours. Then the technician worked on the job for $5 + 2 = 7$ hours. (When using this method you should write out every computation so that you can see later what you did.) The amount billed for the engineer is her hourly rate times the number of hours she worked: $90(5) = 450$. Similarly, the amount billed for the the technician is $60(5 + 2) = 420$, and the total amount billed is

$$90(5) + 60(5 + 2) = 870 \tag{1}$$

dollars. Obviously our guess was too high!

Now we replace our guess of 5 hours for the amount the engineer worked with the amount she really did work on the project; since we don't know what it is we'll call it x. We then go back to (1) and replace 5 with x everywhere, and the total amount billed with the correct value, $\$645$. This gives us the equation

$$90x + 60(x + 2) = 645$$

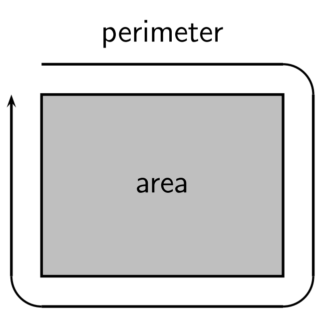
Solving this for x gives us $x = 3.5$ hours, so the engineer worked on the project for 3.5 hours, and the technician worked on it for $3.5 + 2 = 5.5$ hours.

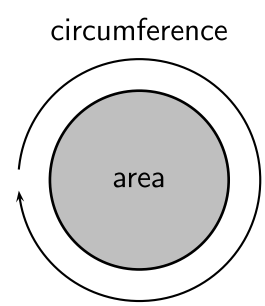
Answers to all word problems MUST include units!

We'll now discuss some specific mathematics for various kinds of problems that we will solve.

Ideas and Formulas From Geometry

We will be working three geometric shapes: rectangles, triangles and circles. When we discus the amount of surface that a shape covers, we are talking about its **area**. Some standard sorts of practical uses of areas are for measuring the amount of carpet needed to cover a floor, or the amount of paint to paint a certain amount of surface. The distance around a shape is usually called its **perimeter**, except in the case of a circle. The distance around a circle is called its **circumference**.



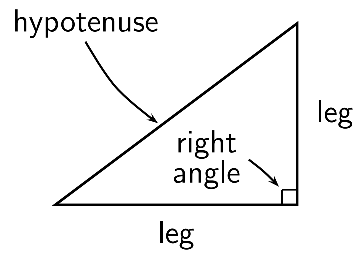


When computing the circumference or area of a circle it is necessary to use a special number called **pi**. Pi is a little over 3, but in decimal form it never ends or repeats. We use the symbol π for it, and its "exact" value is

$$\pi = 3.141592654...$$

Sometimes we round this to 3.14, but to use it most accurately one should use more places past the decimal. We might not care to type in more, but we don't need to - pi is so important that it has its own key on our calculators! Find it on yours.

There is a special kind of triangle that comes up often in applications, called a **right triangle**. A right triangle is a triangle with one of its angles being a right angle ("square corner"). The side opposite the right angle is called the **hypotenuse** of the triangle, and the other two sides are called the **legs** of the triangle. There is a special relationship between the sides of the right triangle; this relationship is expressed by a famous "rule" called the **Pythagorean Theorem**.



A **formula** is an equation that describes the relationship between several (two or more) values. Here are some important geometry formulas:

Geometry Formulas

- For a rectangle with width w and length l, the perimeter is $P = 2w + 2l$ and the area is $A = lw$.
- For a circle with radius r, the circumference is $C = 2\pi r$ and the area is $A = \pi r^2$.
- **Pythagorean Theorem:** If a right triangle has legs with lengths a and b and hypotenuse of length c, then

$$a^2 + b^2 = c^2$$

Let's look at some examples that use these formulas.

◇ **Example 1.3(b):** One of the legs of a right triangle is 8 inches longer than the other leg and the hypotenuse is 16.8 inches long. How long are the legs, to the nearest tenth of an inch?

Solution: For this problem we actually know the equation, $a^2 + b^2 = c^2$, and we know that $c = 16.8$. If the shorter missing leg has a length of x inches, then the longer leg's length is $x + 8$ inches. We must then solve

$$\begin{aligned} x^2 + (x+8)^2 &= 16.8^2 \\ x^2 + x^2 + 16x + 64 &= 282.24 \\ 2x^2 + 16x - 218.24 &= 0 \end{aligned}$$

To solve this we use the quadratic formula:

$$x = \frac{-16 \pm \sqrt{16^2 - 4(2)(-218.24)}}{2(2)} = \frac{16 \pm \sqrt{2001.92}}{4} = -15.2,\ 7.2$$

Because x cannot be negative, the length of one leg is 7.2 inches and the length of the other is $7.2 + 8 = 15.2$ inches.

◇ **Example 1.3(c):** The longest side of a triangle is five centimeters more than the medium side, and the shortest side is half the medium side. The perimeter is $27\frac{1}{2}$ centimeters. Find the lengths of the sides of the triangle.

Solution: We are asked to find the lengths of all the sides of the triangle; let's pick a value for one of the sides and see what happens. Since both the short side and long side are described in terms of the medium side, let's guess the length of the medium side is 10 centimeters. The longest side is 5 centimeters more than that, or $10+5=15$ centimeters, and the shortest side is half the medium, or $\frac{1}{2}(10)=5$ centimeters. The perimeter is then the sum of the three sides:

$$\text{perimeter} = 10 + (10+5) + \frac{1}{2}(10) = 10 + 15 + 5 = 30 \text{ centimeters}$$

Our guess was just a little high, but let's write an equation by replacing 10 everywhere in the above equation with x, and using the desired perimeter:

$$x + (x+5) + \frac{1}{2}x = 27\frac{1}{2}$$

If we first multiply both sides of this by 2 to eliminate the fractions we get

$$2x + 2(x+5) + x = 55,$$

which is easily solved to get $x = 9$ centimeters, the length of the middle side. The length of the longer side is then $9+5=14$ centimeters, and the length of the shorter side is $\frac{1}{2}(9) = 4\frac{1}{2}$ centimeters.

Percents

Many quantities of interest are obtained by taking a certain percentage of other quantities. For example, some salespeople earn a certain percentage of their sales. (In this case what they earn is called their **commission**.) Various taxes are computes by a percentage of the item bought (or owned, in the case of property tax).

Percent means "out of one hundred," so seven percent means seven out of 100. If a person bought something for $200 and they had to pay 7% sales tax, they would pay seven dollars for every $100, or $14. When doing math with a percent, it is necessary to change a percent to its decimal form. Since seven percent means seven out of one hundred,

$$7\% = \frac{7}{100} = 0.07$$

This last value is called the **decimal form of the percent**. Note that the decimal in $7 = 7.0$ has been moved two places to the left.

◇ **Example 1.3(d):** Change 4.25% to decimal form.

Solution: Moving the decimal two places to the left gives us $4.5\% = 0.045.$

Working With Percents

To find p percent of an amount A, change p to decimal form and multiply it times A.

◇ **Example 1.3(e):** The standard tip for waiters and waitresses is 15-20% of the cost of the meal. If you and some friends go out to eat and the total bill is \$87.50, how much of a tip should you give your waitress if you wish to give 15%?

Solution: To find 15% of \$87.50 we change the percentage to decimal form and multiply it by the amount: $0.15(87.50) = 13.13$. A 15% tip on \$87.50 tip would be \$13.13.

Note that answers to questions about money *should always be rounded to the nearest cent*, unless told otherwise.

◇ **Example 1.3(f):** A salesperson in a jewelry store gets a *commission* of 15% on all their sales. If their total in commissions for one month is \$2602.43, what were her total sales for the month?

Solution: This Example is a little more challenging than the last one. Let's suppose that her sales were \$10,000. Then, in the same manner as the last example, her commission would be $0.15(10,000) = 1500$. This is too low, but it allows us to see how to solve the problem. If we let her true sales be s and modify our computation, we arrive at the equation $0.15s = 2602.43$. Solving this, we see that her sales for the month were \$17,349.53.

◇ **Example 1.3(g):** The sales tax in a particular city is 4.5%. You buy something there for \$78.37, including tax. What was the price of the thing you bought?

Solution: We again begin with a guess. Suppose the price of the item was \$70. Then the tax alone would be $0.045(70) = 3.15$ and the total cost would be $70+0.045(70) = 73.15$. Now, letting the actual price be p we get the equation $p + 0.045p = 78.37$. This is solved like this:

$$\begin{aligned} 1p + 0.045p &= 78.37 \\ 1.045p &= 78.37 \\ p = 75.00 & \end{aligned}$$

The price of the item was \$75.00.

This last example illustrates an important point. The actual value obtained when solving the equation was 74.99521531... When rounding to two place past the zero, the 9 in the hundredth's place changes to ten, making it a zero, which in turn causes the next two places to round up. You might then be tempted to give the answer as just \$75, which is the correct amount. However, a person looking at this answer and not knowing how it was rounded might think that the value before rounding was between 74.95 and 75.04 when it was, in fact, far more accurate than that. Including the two zeros indicates that the price was $75 even when rounded to the nearest cent.

Later we will look at compound interest, which is how banks truly calculate interest. For now we'll use the following simpler formula for computing interest.

Simple Interest

Suppose that a **principal** of P dollars is invested or borrowed at an annual interest rate of r percent (in decimal form) for t years. The amount A that is then had or owed at the end of t years is found by

$$A = P + Prt$$

◇ **Example 1.3(h):** You borrow $1000 at 8.5% interest for five years. How much money do you owe at the end of the five years?

Solution: Here we have $P = 1000$, $r = 0.085$ and $t = 5$, giving

$$A = 1000 + 1000(0.085)(5) = 1425$$

The amount owed after five years is $1425.00

◇ **Example 1.3(i):** You are going to invest $400 for 5 years, and you would like to have $500 at the end of the five years. What percentage rate would you need to have, to the nearest hundredth of a percent?

Solution: In this case we know that $P = 400$, $t = 5$ and $A = 500$. We substitute these values and solve for r:

$$\begin{aligned} 500 &= 400 + 400(5)r \\ 100 &= 2000r \\ 0.05 &= r \end{aligned}$$

You would need a rate of 5% to have $400 grow to $500 in five years.

Solving Formulas

Suppose that we wanted to use the simple interest formula $A = P + Prt$ to answer each of the following questions:

1) How long must $300 be invested for at 4.5% in order to have $500?

2) How long must $2000 be invested for at 7.25% in order to have $4000?

3) How long must \$800 be invested for at 5% in order to have \$1000?

To answer the first of these, we would substitute the known values and solve for t:

$$\begin{aligned} 500 &= 300 + 300(0.045)t \\ 200 &= 13.5t \\ 14.8 &= t \end{aligned}$$

It would take 14.8 years for \$300 to grow to \$500 at an interest rate of 4.5%. It should be clear that the other two questions would be answered in exactly the same way. In situations like this, where we might use the same formula over and over, always solving for the same thing, it is usually more efficient to solve the equation *BEFORE* substituting values into it.

◇ **Example 1.3(j):** Solve $A = P + Prt$ for t.

Solution:

$$\begin{aligned} A &= P + Prt \\ A - P &= Prt \\ \frac{A-P}{Pr} &= t \end{aligned}$$

We will call this "solving a formula"; other people might call it other things, but there is not real good name for it. Anyway, if you have trouble doing this you might consider trying the following procedure:

- *If the unknown you are solving for only occurs once in the formula:* Imagine what operations you would do, and in what order, if you knew its value and the values of all other variables and parameters. Then start "undoing" those operations one by one, *in the reverse order that you would do them if you had numbers for everything.*
- *If the unknown that you are trying to solve for occurs to the first power in more than one place:* Get all terms containing it on one side of the equation (and all other terms on the other side) and factor it out. Then divide by whatever it is multiplied by.

Consider again solving for t in the formula $A = P + Prt$. If we knew numbers for all values, on the right side we would multiply t by P and r, then add P. The above tells us that in solving for t we would first "undo" adding P by subtracting it from each side. Then we would undo multiplying by P and r by dividing both sides (the entire side in each case) by Pr. You can see those steps carried out in the example.

Now let's look at the situation addressed by the second bullet above.

◇ **Example 1.3(k):** Solve $ax - 8 = bx + c$ for x.

Solution: Here the unknown that we are solving for, x, occurs (to the first power) in two places. We'll solve for it by following the procedure from the second bullet above:

$$\begin{aligned} ax - 8 &= bx + c && \text{the original equation} \\ ax - bx &= c + 8 && \text{get all terms with } x \text{ on one side,} \\ & && \text{terms without } x \text{ on the other} \\ (a-b)x &= c + 8 && \text{factor } x \text{ out of the left side} \\ x &= \frac{c+8}{a-b} && \text{divide both sides by } a - b \end{aligned}$$

If the unknown to be solved for occurs in more than one place, and not all to the first power, then there will be no standard method for solving for it.

We will now look at an example of a computation that we will find valuable later.

◇ **Example 1.3(l):** Solve $-2x-7y=14$ for y. Give your answer in $y=mx+b$ form.

Solution:

$$\begin{aligned} -2x-7y &= 14 \\ -7y &= 2x+14 \\ y &= \frac{2x+14}{-7} \\ y &= \frac{2x}{-7}+\frac{14}{-7} \\ y &= -\tfrac{2}{7}x-2 \end{aligned}$$

You probably recognize the result as the $y=mx+b$ form of the equation of a line.

Section 1.3 Exercises To Solutions

Use an equation to solve each problem, whether it is one that has been given or one that you create yourself.

1. The hypotenuse of a right triangle has length 13 inches and one of the legs has length 12 inches. What is the length of the other leg?

2. The circumference of a circle is 16.4 inches. Find the radius of the circle, to the nearest tenth of an inch.

3. Sales tax in a particular city is 5.5%, and you buy an item with a *pre-tax* price of $19.95.

   (a) How much sales tax will you have to pay for the item?

   (b) How much will you have to pay for the item, including tax?

   (c) Suppose that, in the same city, you buy something else, and your total cost *with tax* is $87.48. What was the price of the item?

4. A salesperson in an art gallery gets a monthly salary of $1000 plus 3% of all sales over $50,000. How much do they make in a month that they sell $112,350 worth of art?

5. Suppose that the owner of the gallery from the previous exercise changes the pay structure to the following: A salesperson gets a monthly salary of $1000 plus 2% of all sales between $50,000 and $100,000, and 5% of all sales over $100,000.

   (a) What is the monthly pay for a salesperson in a month that they have $82,000 of sales?

   (b) What is the monthly pay for a salesperson in a month that they have $127,000 of sales?

6. You borrow some money at an interest rate of 8.5%, simple interest. After four years it costs you $1790.25 to pay off the loan. How much did you borrow?

7. The length of a rectangle is one more than three times the width, and the area is 520. Write an equation and use it to find the length and width of the rectangle.

8. Sales tax in a state is 5.5%. If the tax on an item was $10.42, what was the price of the item?

9. One leg of a right triangle is twice as long as the other leg, and the hypotenuse has a length of 15 feet. How long are the legs?

10. After a 10% raise your hourly wage is $7.37 per hour. What was it before the raise?

11. The length of a rectangle is three more than twice the width. The area is 44. Find the length and width.

12. The sales tax in a particular city is 6.5%. You pay $41.37 for an electric can opener, including tax. What was the price of the can opener?

13. The length of the shortest side of a triangle is half the length of the longest side. The middle side is 1.4 inches longer than the shortest side, and the perimeter is 43.4 inches. Find the lengths of all three sides.

14. A retailer adds 40% of her cost for an item to get the price she sells it at in her store. Find her cost for an item that she sells for $59.95.

15. The hypotenuse of a right triangle is two more than one of the legs, and the other leg has length 8. What are the lengths of the sides of the triangle?

16. (a) A rectangle has a width of 7 inches and a length of 10 inches. Find the area of the rectangle.
    (b) Suppose that the length and width of the rectangle from part (a) are each doubled. What is the new area?
    (c) When the length and width are doubled, does the area double as well?
    (d) Suppose that the length and width are each multiplied by three, then the area is computed. How many times bigger than the original area do you think the new area will be? Check your answer by computing the new area - were you correct?

17. You invest $1200 at 4.5% simple interest.
    (a) How much money will you have if you take it out after 4 years?
    (b) How many years, to the nearest tenth, would it take to "double your money?"

18. Solve the equation $A = P + Prt$ for r.

19. Solve $PV = nRT$ for R.

20. Solve $ax + 3 = bx - 5$ for b.

21. The formula $F = \frac{9}{5}C + 32$ is used for converting a Celsius temperature C to a Fahrenheit temperature F.

   (a) Convert 25° Celsius to Fahrenheit.

   (b) Even though it doesn't work quite as well, the formula can also be used to convert from Fahrenheit to Celsius. Convert 95° Fahrenheit to Celsius.

   (c) Solve the formula $F = \dfrac{9}{5}C + 32$ for C.

   (d) Use your result from part (c) to convert 77°F (this means 77 degrees Fahrenheit) to Celsius.

22. Solve $P = 2w + 2l$ for l.

23. Solve $C = 2\pi r$ for r.

24. Solve each equation for y. Give your answers in $y = mx + b$ form.

   (a) $3x + 4y = -8$ (b) $5x + 2y = -10$ (c) $3x - 2y = -5$

   (d) $3x - 4y = 8$ (e) $3x + 2y = 5$ (f) $-5x + 3y = 9$

25. Solve $8x + 3 = 5x - 7$ for x.

26. Solve each equation for x.

   (a) $ax + b = cx + d$ (b) $x + 1.065x = 8.99$

   (c) $ax + 3 = bx - 5$ (d) $ax + 7 = b(x + c)$

27. Solve $A = P + Prt$ for P.

28. Read Example 1.3(a) again, where the equation was determined to be

$$90x + 60(x + 2) = 645.$$

   For each of the following, write a brief sentence telling *in words* what the given part of the equation represents.

   (a) x (b) $90x$ (c) $x + 2$ (d) $60(x + 2)$ (e) $90x + 60(x + 2)$

1.4 Equations in Two Unknowns

Performance Criteria:

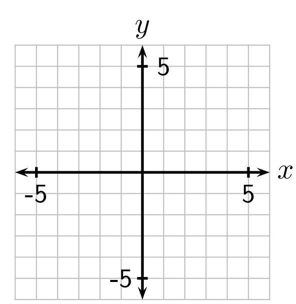
1. (f) Find solution pairs for an equation in two unknowns.
   (g) Graph the solution set of an equation in two unknowns without using a graphing calculator.
   (h) Identify x- and y-intercepts of an equation in two unknowns from its graph; find x- and y-intercepts of an equation in two unknowns algebraically.

When working with equations in one unknown, our goal has always been to *solve* them. This means we are looking for all values (called **solutions**) of the unknown that make the equation true. Consider the equation $2x+y=5$; how do we "solve" an equation like this? Well, it would seem reasonable to say that a **solution of an equation in two unknowns** is a *pair* of numbers, one for each of the unknowns, that makes the equation true. For the equation $2x+y=5$, the pair $x=2,\ y=1$ is a solution. So are the pairs $x=0,\ y=5$, $x=3,\ y=-1$ and $x=\frac{1}{2},\ y=4$. At this point you probably realize that there are infinitely many such pairs for this one equation! Since there is no hope of listing them all, we will instead draw a picture that describes all of them, to some extent.

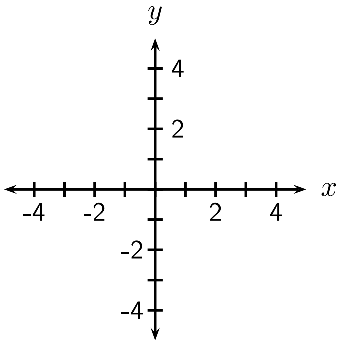
In order to write a bit less, we usually write $(2,1)$ instead of $x=2,\ y=1$. $(2,1)$ is called an **ordered pair**, and it is understood that the first value is for x and the second for y. Finding such pairs is easier if we can get x or y alone on one side of the equation. The equation $y=-2x+5$ is equivalent to $2x+y=5$, but instead of trying to think about what values of x and y make $2x+y=5$ true, we can substitute any value we wish for x into $y=-2x+5$ to find the corresponding value for y. We usually keep track of our results in a table like the one shown to the right, with which you are probably familiar. Note that when displaying solution pairs in such a table they are not necessarily shown in any particular order.

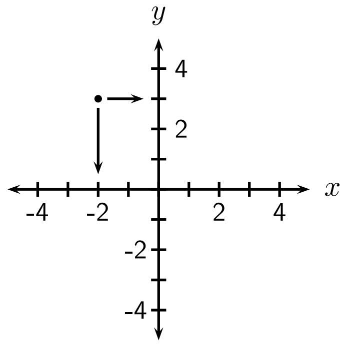
x	y
2	1
0	5
3	-1
$\frac{1}{2}$	4

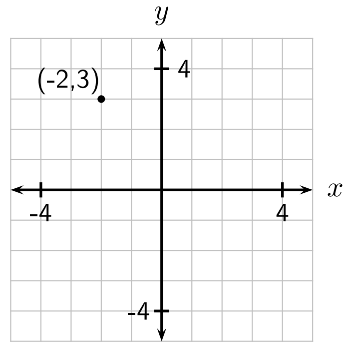
Since there is no hope of listing all of the solutions, we instead draw a picture, or graph, that gives an indication of what all those pairs are. The basic idea is that every ordered pair of numbers corresponds to a point on a grid like the one shown to the right (with the understanding that the grid continues infinitely far in all directions). This grid is called the coordinate plane, or **Cartesian plane** after the mathematician and philosopher René Descartes. The darkened horizontal line is called the x-**axis** and the darkened vertical line is the y-**axis**.



In some sense the coordinate plane is simply two number lines set at right angles to each other, as shown to the left at the top of the next page. Given a point in the plane, we travel to the x-axis in a direction parallel to the y-axis to get a number from the x-number line, then travel from the point to the y-axis in a direction parallel to the x-axis to get a second number. These two numbers are called the **coordinates** of the point. The point shown in the middle picture at the top of the next page has coordinates $x = -2$, $y = 3$ or, as an ordered pair, $(-2, 3)$. The last picture, on the right, shows how easy it is to determine the coordinates of a point when we cover the plane with a grid.

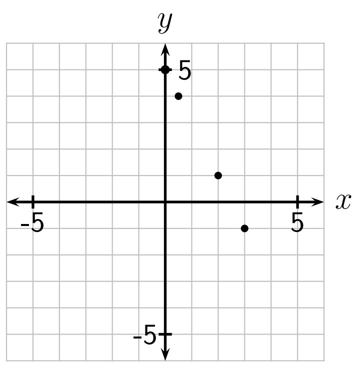


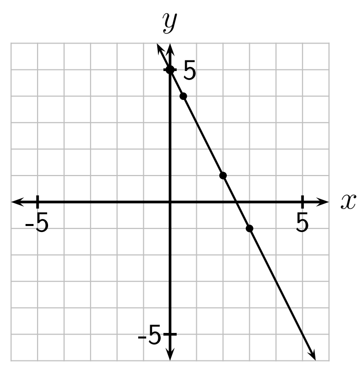




So for an equation containing two unknowns, like $2x + y = 5$, we find enough solution pairs that when we plot a point for each of them in the coordinate plane we can determine where all the others that we haven't found are. We then draw in a line or a curve, every point of which represents a solutions pair. In the previous section we found the solution pairs given in the table on the left below for the equation $y = -2x + 5$, which is equivalent to $2x + y = 5$. The corresponding points have been plotted on the grid in the middle, and we can see that they appear to lie on a line. That line is shown on the grid to the right below.

x	y
2	1
0	5
3	-1
$\frac{1}{2}$	4





Note the use of arrowheads to indicate that the line continues in both directions off the edges of the graph.

In this course we will usually graph solutions to equations having just y on the left side and an expression of x on the right side. Examples of such equations are

$$y = \tfrac{1}{2}x^2 - 2x - 1 \qquad\qquad y = \sqrt{x + 3} \qquad\qquad y = |x + 1| - 5$$

For now the process will be as follows:

1) Choose (sometimes carefully) values of x and substitute them into the right side of the equation to get corresponding values of y, recording the values in a table as shown above.

2) Plot each pair of values as a point on a coordinate grid. If any appear to not fit a pattern, check the computation of the 'y value.

3) Draw a line or curve through the points you have plotted. Extend it and add arrowheads when appropriate, to indicate that the graph extends beyond the grid.

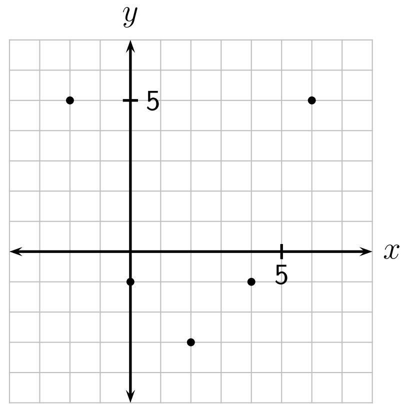
When we carry out this process we are technically graphing the solution set of the equation, but in a slight abuse of language we will say that we are "graphing the equation."

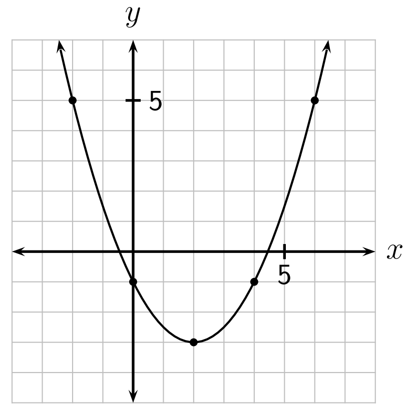
Let's see how this process is carried out for the first equation above.

◇ **Example 1.4(a):** Graph the equation $y = \frac{1}{2}x^2 - 2x - 1$.

Solution: We must choose some values of x for which to find the corresponding y values. A good first choice of x is $x = 0$, resulting in $y = 1$. Often we would choose $x = 1$ next, but we can see that results in a fraction for y, so let's use $x = 2$ instead, resulting in $y = -3$. Continuing like this results in the table of values below and to the left, which are plotted on the left grid below. On the right grid below we connect the dots and extend the curve and add arrowheads on the ends to indicate that the graph keeps going.

x	y
-2	5
0	-1
2	-3
4	-1
6	5

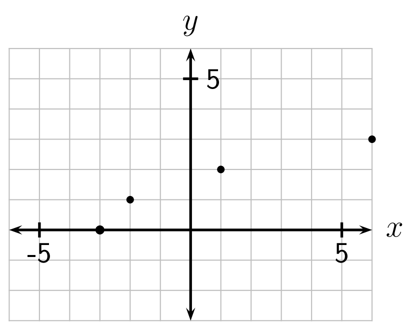


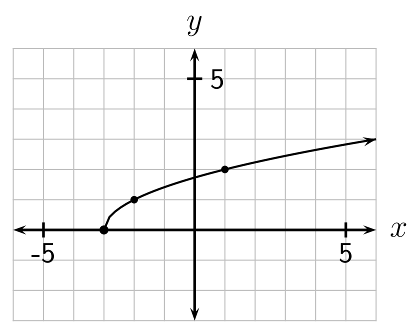


◇ **Example 1.4(b):** Graph the equation $y = \sqrt{x+3}$.

Solution: Here our y values will be "nicer" (that is, will be integers) if we choose values of x such that $x+3$ is a perfect square. Recalling that the perfect squares are $0, 1, 4, 9, ...,$ we can see that when $x = -3$, $x + 3 = 0$ and we get $y = \sqrt{-3+3} = \sqrt{0} = 0$. Similarly, $x = -2$ gives $y = \sqrt{-2+3} = \sqrt{1} = 1$, $x = 1$ gives $y = \sqrt{1+3} = \sqrt{4} = 2$, $x = 6$ gives $y = \sqrt{6+3} = \sqrt{9} = 3$ and $x = 13$ gives $y = \sqrt{13+3} = \sqrt{16} = 4$. These values can be organized into the table shown to the left below, and the corresponding points are plotted on the left grid below. In this case the points appear to lie on half of a parabola, as plotted on the right grid. There are no points to the left of $-3, 0)$ because the square root is undefined for x values less than -3, so there is no arrowhead on that end of the curve.

x	y
-3	0
-2	1
1	2
6	3



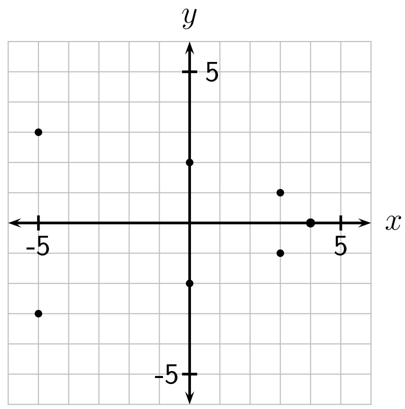


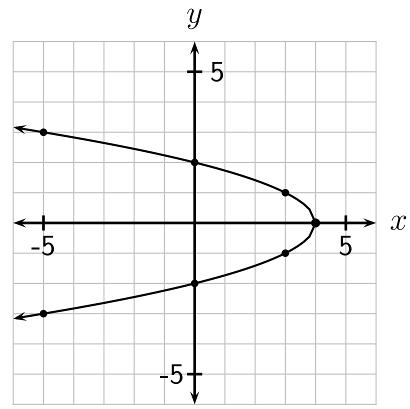
Both examples that we have looked at so far have had y alone on one side. We now look at an example with x alone on one side. You may feel some compulsion to solve for y, but *there is no need to, and if not done carefully it can lead to incorrect results!*

◇ **Example 1.4(c):** Graph the equation $x = 4 - y^2$.

Solution: In this case we should *choose values for* y *instead of* x, and we can see that when $y = 0$ we get $x = 4$. We should also note that we are squaring y and subtracting the result from five to get x, so we will get the same thing for a negative value as we do for the corresponding positive value. For example, when $y = 1$ we get $x = 4 - (1)^2 = 3$, and when $y = -1$, $x = 4 - (-1)^2 = 3$. Letting $y = 1, 2, 3$ and their negatives gives us the values in the table below and to the left. The corresponding points are plotted on the left grid below, and the solution curve is graphed on the right grid.

x	y
-5	-3
0	-2
3	-1
4	0
3	1
0	2
-5	3

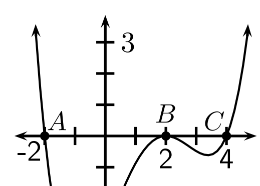




One goal we have is to be able to graph equations quickly,efficiently, and accurately. In later sections you will see that doing so requires first identifying what we expect a graph to look like, then plotting key points, rather than just randomly finding and plotting points.

x- and y-Intercepts

Consider the graph shown to the right, for some unknown equation. Points A, B and C are called x-**intercepts** of the graph or equation, and D is a y-**intercept**. *It is possible to have more than one intercept on each axis.* The points are called intercepts because they are where the graph of the equation intercepts the axes. When asked for the intercepts, we mean to give their coordinates, so in this case we would say that the x-intercepts are $(-2, 0)$, $(2, 0)$ and $(4, 0)$. The y-intercept is $(0, -3)$.



We should observe that the y-coordinate of every x-intercept is zero, and the x-coordinate of the y-intercept (and any other y-intercept, if there were more) is zero. This indicates the following.

Finding Intercepts

- To find the x-intercepts for an equation, let $y = 0$ and solve for x.
- To find the y-intercepts for an equation, let $x = 0$ and solve for y.
- We sometimes give the intercepts as single numbers rather than ordered pairs, because we know that the y-coordinates of x-intercepts must be zero, and vice-versa.

From the last bullet, we can give the x-intercepts for the graph on the previous page as -2, 2 and 4, and the y-intercept can be given as just -3. More often you will be asked to find the intercepts for a given equation, like in the next example.

◇ **Example 1.4(d):** Find the intercepts of $x = 4 - y^2$

Solution: To find the x-intercepts we set $y = 0$ and solve for x, giving $x = 4$. To find the y-intercepts we set $x = 0$ and solve, giving $y = 2$ or $y = -2$. Therefore the x-intercept of $x = 4 - y^2$ is 4 and the y-intercepts are 2 and -2.

Section 1.4 Exercises To Solutions

Check your answers to all graphing exercises by graphing the equation on a graphing calculator, or with an online grapher like *www.desmos.com*.

1. In this exercise you will graph the equation $y = x^2 - 2x - 2$; the graph will be a parabola similar to the one in Example 1.4(a).

   (a) Begin by letting $x = 0, 1, 2, ...$ and continuing finding the corresponding y values until you obtain values too large to fit on a reasonable sized graph. Put your values in a table.

   (b) Find y values corresponding to some negative x values until those y values are also too large to fit on a reasonable sized graph,adding these solution pairs to your table.

   (c) Plot your points. If any points look out of place, go back and check your calculation of the y. Once you see the points as being on a parabola, draw the parabola. Indicate that the ends keep going by putting arrowheads on them.

   (d) Check your answer as described above.

2. The graph of the equation $y = -x^2 - 4x$ is also a parabola. follow the same procedure as outlined in Exercise 1 to determine its graph.

3. Consider the equation $y = -\frac{3}{2}x + 3$.

   (a) Note that if x is an even number, $-\frac{3}{2}x$ will be an integer. Calculate y values for $x = -2,\ 0,\ 2,\ 4$ and put your results in a table.

   (b) Plot the points listed in your table. They should lie on a (straight) line - if they don't, check your calculations.

   (c) Draw in the line, again with arrowheads to indicate it keeps going. Check your answer, again using a graphing calculator or online grapher.

4. Consider the equation $y = \sqrt{5-x} - 1$

   (a) Note that $\sqrt{5-x}$ is undefined for $x = 10,$ so we would not be able to find a y value corresponding to that value of x. What is the largest value that you can use for x? What is y in that case?

   (b) Choose three more values of x for which $5-x$ has the values of $1, 4$ and $9,$ and find the corresponding y values. Put those pairs, and the pair from (a), in a table.

   (c) The graph in this case should be *half of* a parabola. Plot your points, sketch the graph and check it.

5. You should know the graphs of the following equations, or be able to create them very quickly. For each, (*i*) make a table of solution pairs for $x = -2,\ -1,\ 0,\ 1,\ 2,$ (*ii*) plot the corresponding points, (*iii*) sketch the graph and (*iv*) check your graph.

   (a) $y = x$ (b) $y = x^2$ (c) $y = |x|$ (d) $y = x^3$

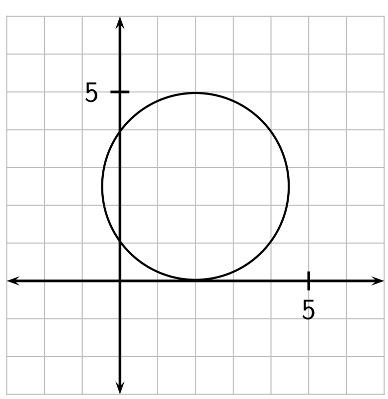
6. You should also know the graphs of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$. Note that we cannot find square roots of negative numbers, but we *CAN* find cube (third) roots of negative numbers.

   (a) Remembering that zero and one are both perfect squares, make a table of x and y values for the first four perfect square values of x. Plot the first three pairs, keep in mind where the fourth pair would be, and graph the equation. Check your graph.

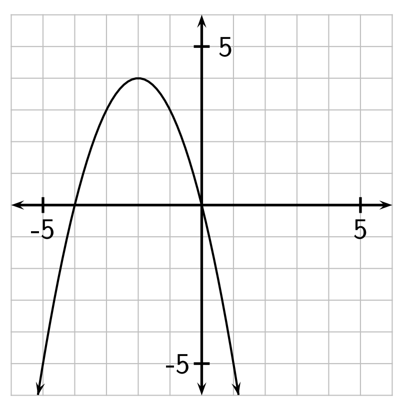
   (b) For the equation $y = \sqrt[3]{x}$, find y values for $x = -2,\ -1,\ 0,\ 1,\ 2$ and plot the resulting points. Sketch the graph and check it.

7. Give the x- and y-intercepts for each of the graphs below, indicating clearly which is which.

   (a)



   (b)



8. Give the x- and y-intercepts for the graphs in each of the following examples. You will have to approximate some of them - use decimals to the nearest tenth.

   (a) Example 1.4(c). (b) Example 1.4(a). (c) Example 1.4(b).

9. Give the intercepts of the line graphed on page 26.

10. Find the intercepts of $-5x + 3y = 15$, being sure to indicate which are x-intercepts and which are y-intercepts.

11. Find the intercepts of $y = x^2 - 2x - 3$.

12. This exercise will reinforce the fact that intercepts are not always integers! (Remember that integers are positive or negative whole numbers, or zero.) Find the intercepts of the equation $2x + 3y = 9$.

13. This exercise illustrates that the an x-intercept can also be a y-intercept.

    (a) Find the x-intercepts of $y = x^2 - 5x$. *Give your answers as ordered pairs.*

    (b) Find the y-intercepts of $y = x^2 - 5x$. *Give your answers as ordered pairs.*

14. Find the intercepts for each of the following equations algebraically.

    (a) $x = y^2 - 2y$ (b) $y = \sqrt{x+4}$ (c) $x = 2y - 3$

    (d) $y = (x+3)^2$ (e) $x^2 + y = 1$ (f) $y = x^3$

    (g) $3x + 5y = 30$ (h) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ (i) $\sqrt{x+4} = y + 1$

    (j) $y = |x - 3| + 1$ (k) $y = |x + 2|$

15. Find the x- and y-intercepts of the following equations algebraically. Note that for (c) and (d) you are given two forms of the same equation. One will be easier to use to find x-intercepts and the other will be easier to use for finding y-intercepts. (**Hint for (c) and (d):** Recall the result of Exercise 5 in Section 1.2.)

    (a) $y = \frac{1}{3}(x+3)^2(x-1)(x-2)$

    (b) $y = -\frac{3}{5}(x-5)(x+2)(x+4)$

    (c) $y = \dfrac{2x^2 - 8}{x^2 - 2x - 3} = \dfrac{2(x+2)(x-2)}{(x-3)(x+1)}$

    (d) $y = \dfrac{3x}{x^2 - 2x + 1} = \dfrac{3x}{(x-1)^2}$

1.5 Equations in Two Unknowns: Lines

Performance Criterion:

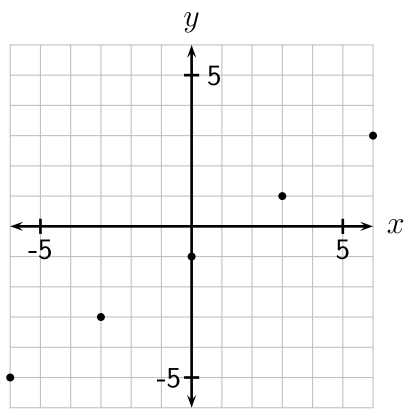
1. (i) Graph a line; determine the equation of a line.

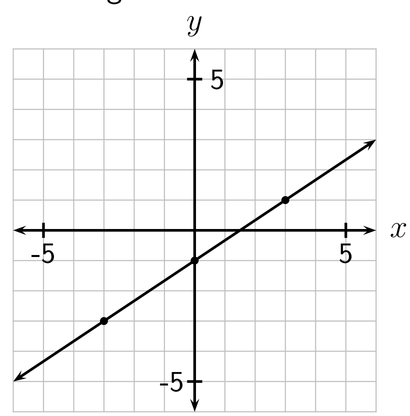
Let's begin with an example.

◇ **Example 1.5(a):** Graph the equation $2x - 3y = 3$.

Solution: If we solve the equation for y we obtain $y = \frac{2}{3}x - 1$. When $x = 0$ we get $y = -1$, and we can avoid fractional values of y if we let x be various multiples of three. The table below and to the left shows some solution pairs obtained in this way, and the corresponding points are plotted on the left graph below. We can then see that the graph of the equation is the line shown below and to the right.

x	y
-6	-5
-3	-3
0	-1
3	1
6	3





It should be clear that any equation of the form $Ax + By = C$ can be put in the form $y = mx + b$, as we did above, and the graph of such an equation is always a line. We will soon see that there is a much faster way to graph a line, that you may recall. To use it we need the idea of the slope of a line.

Slopes of Lines

The slope of a line is essentially its "steepness," expressed as a number. Here are some facts about slopes of lines that you may recall:

- The slope of a horizontal line is zero, and the slope of a vertical line is undefined. (Sometimes, instead of "undefined," we say a vertical line has "no slope," which is not the same the same as a slope of zero!)

- Lines sloping "uphill" from left to right have positive slopes. Lines sloping downhill from left to right have negative slopes.

- The steeper a line is, the larger the *absolute value* of its slope will be.

- To find the numerical value of the slope of a line we compute the "rise over the run." More precisely, this is the amount of vertical distance between two points on the line (*any* two points will do) divided by the horizontal distance between the same two points. If the two points are (x_1, y_1) and (x_2, y_2), the slope is given by the formula

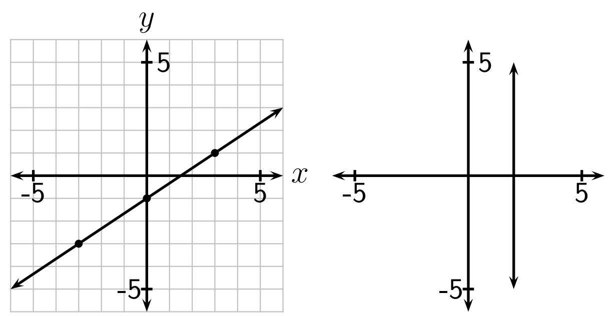
$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

where m represents the slope of the line. It is standard in mathematics to use the letter m for slopes of lines.

- Parallel lines have the same slope. Perpendicular lines have slopes that are negative reciprocals; for example, lines with slopes $m_1 = -\frac{2}{3}$ and $m_2 = \frac{3}{2}$ are perpendicular. (The subscripts here designate one slope from the other.)

Let's look at some examples involving the slopes of lines.

◇ **Example 1.5(b):** Give the slope of the lines with the graphs shown below.



Line 1 Line 2

Solution: For Line 1 we first note that it slopes uphill from left to right, so its slope is positive. Next we need to determine the rise and run between two points; consider the points $(-3, -3)$ and $(3, 1)$. Starting at $(-3, -3)$ and going straight up, we must go up four units to be level with the second point, so there is a rise of 4 units. We must then "run" six units to the right to get to $(3, 1)$, so the run is 6. We already determined that the slope is positive, so we have $m = \frac{\text{rise}}{\text{run}} = \frac{4}{6} = \frac{2}{3}$.

Suppose we pick any two points on Line 2. The rise will be some number that will vary depending on which two points we choose. However, the run will be zero, regardless of which points we choose. Thus when we try to find the rise over the run we will get a fraction with zero in its denominator. Therefore the slope of Line 2 is undefined.

◇ **Example 1.5(c):** Give the slope of the line through $(-3, -4)$ and $(6, 8)$.

Solution: Here we will let the first point given be (x_1, y_1), and the second point be (x_2, y_2). We then find the rise as $y_2 - y_1$ and the run as $x_2 - x_1$, and divide the rise by the run:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - (-4)}{6 - (-3)} = \frac{12}{9} = \frac{4}{3}.$$

◇ **Example 1.5(d):** The slope of Line 1 is $-\frac{3}{5}$. Line 2 is perpendicular to Line 1 and Line 3 is parallel to Line 1. What are the slopes of Line 2 and Line 3?

Solution: Because Line 2 is perpendicular to Line 1, its slope must be the negative reciprocal of $-\frac{3}{5}$, or $\frac{5}{3}$. Line 3 is parallel to Line 1, so it must have the same slope, $-\frac{3}{5}$.

◇ **Example 1.5(e):** The slope of Line 1 is 0. If Line 2 is perpendicular to Line 1, what is its slope?

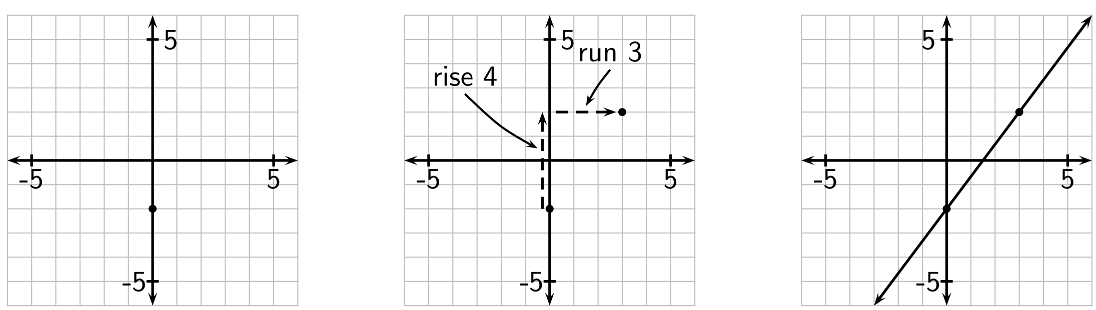
Solution: Because its slope is zero, Line 1 is horizontal. A perpendicular line is therefore vertical, so its slope is undefined.

Equations of Lines

Note that Line 1 from Example 1.5(b) is the graph of $y = \frac{2}{3}x - 1$ obtained in Example 1.5(a). This illustrates that the graph of any equation of the form $y = mx + b$ is a line with slope m. Furthermore, when $x = 0$ the equation $y = mx + b$ yields $y = b$, so b is the y-intercept of the line. This can easily be seen to be true for Example 1.5(a). Let's use these ideas to get another graph.

◇ **Example 1.5(f):** Graph the equation $4x - 3y = 6$.

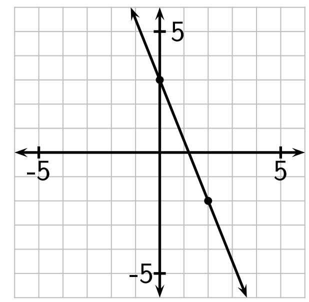
Solution: The equation is of the form $Ax + By = C$, so we know its graph will be a line. Solving for y, we obtain $y = \frac{4}{3}x - 2$. This tells us that the y-intercept is -2. We then plot that point,as shown on the grid to the left below. Next we note that the slope of the line is $\frac{4}{3}$, so we can obtain another point by *starting at* $(0, -2)$ and going up four units and right three units (we would have gone left if the slope was negative) to get a second point. This is shown on the middle grid below. We only need two points to determine the graph of a line, so we can now draw the line, as shown on the grid below and to the right.



We only need two points to determine the graph of a line, but we often get additional points by applying the slope again from the new point obtained, or "backward" from the y-intercept. Had we done that in the above example we would have obtained points located at the tips of the two arrowheads.

◇ **Example 1.5(g):** Give the equation of the line graphed below and to the right.

Solution: Here we begin by noting that the equation of any line can be written in the form $y = mx + b$, and the task essentially boils down to finding m and b. In our case b is the y-intercept 3. Next we use any two points on the line to find the slope of the line, noting that the slope is negative since the line slopes "downhill" from left to right. Using the y-intercept point and the point $(2, -2)$, we see that the vertical change is five units and the horizontal change is two units. Therefore the slope is $m = -\frac{5}{2}$ and the equation of the line is $y = -\frac{5}{2}x + 3$.



Students sometimes give their result from an exercise like the last example as $y = -\frac{5}{2} + 3$. This is the same as $y = \frac{4}{3}$, which *IS* the equation of a line (the horizontal line with y-intercept $\frac{4}{3}$), but it is not the equation of the line graphed above. Be sure to include the x from $y = mx + b$ in your answer to this type of question, or the two that follow.

◇ **Example 1.5(h):** Find the equation of the line through $(-4, 5)$ and $(2, 1)$ **algebraically**.

Solution: Again we want to use the "template" $y = mx + b$ for our line. In this case we first find the slope of the line:

$$m = \frac{5-1}{-4-2} = \frac{4}{-6} = -\frac{2}{3}.$$

At this point we know the equation looks like $y = -\frac{3}{2}x + b$, and we now need to find b. To do this we substitute the values of x and y from *either* ordered pair that we are given into $y = -\frac{2}{3}x + b$ (being careful to get each into the correct place!) and solve for b:

$$\begin{aligned} 1 &= -\tfrac{2}{3}(2) + b \\ \tfrac{3}{3} &= -\tfrac{4}{3} + b \\ \tfrac{7}{3} &= b \end{aligned}$$

We now have the full equation of the line: $y = -\frac{2}{3}x + \frac{7}{3}$.

◇ **Example 1.5(i):** Find the equation of the line containing $(1, -2)$ and perpendicular to the line with equation $5x + 4y = 12$.

Solution: In this case we wish to again find the m and b in the equation $y = mx+b$ of our line, but we only have one point on the line. We'll need to use the other given information, about a second line. First we solve its equation for y to get $y = -\frac{5}{4}x + 3$. Because our line is perpendicular to that line, the slope of our line is $m = \frac{4}{5}$ and its equation is $y = \frac{4}{5}x + b$. As in the previous situation, we then substitute in the given point on our line to find b:

$$\begin{aligned} -2 &= \tfrac{4}{5}(1) + b \\ -\tfrac{10}{5} &= \tfrac{4}{5} + b \\ -\tfrac{14}{5} &= b \end{aligned}$$

The equation of the line is then $y = \frac{4}{5}x - \frac{14}{5}$.

◇ **Example 1.5(j):** Find the equation of the line through $(2, -3)$ and $(2, 1)$.

Solution: This appears to be the same situation as Example 1.5(h), so let's proceed the same way by first finding the slope of our line:

$$m = \frac{-3-1}{2-2} = \frac{-4}{0},$$

which is undefined. Hmmm... what to do? Well, this means that our line is vertical. Every point on the line has a x-coordinate of 2, regardless of its y-coordinate, so the equation of the line is $x = 2$.

This last example points out an idea that we will use a fair amount later:

Equations of Vertical and Horizontal Lines

The equation of a horizontal line through the point (a, b) is $y = b$, and the equation of a horizontal line through the same point is $x = a$.

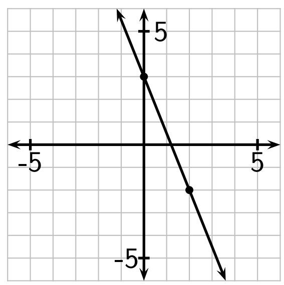
Section 1.5 Exercises To Solutions

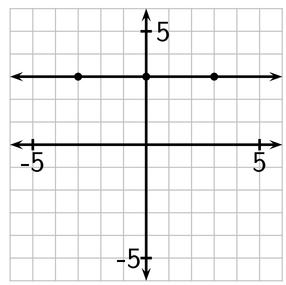
1. Find the slopes of the lines through the following pairs of points.

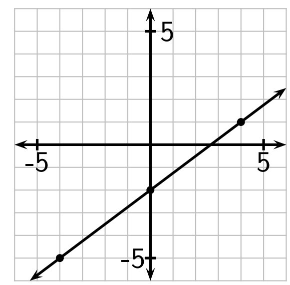
   (a) $(-4, 5)$ and $(2, 1)$ (b) $(-3, 7)$ and $(7, 11)$ (c) $(3, 2)$ and $(-5, 2)$

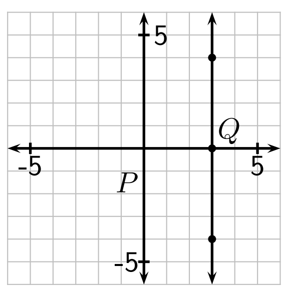
   (d) $(1, 7)$ and $(1, -1)$ (e) $(-4, -5)$ and $(-1, 4)$ (f) $(3, -1)$ and $(-3, 2)$

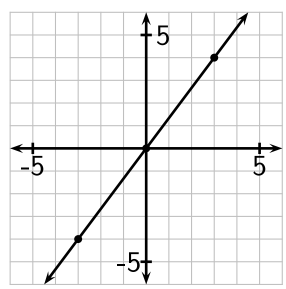
2. Find the slope of each line.

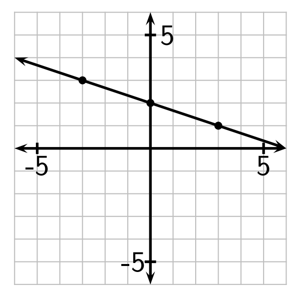
(a)


(b)


(c)


(d)


(e)


(f)


3. Give the equation of each line in the previous exercise.

4. Line 1 has slope $-\frac{1}{3}$, and Line 2 is perpendicular to Line 1. What is the slope of Line 2?

5. Line 1 has slope $\frac{1}{2}$. Line 2 is perpendicular to Line 1, and Line 3 is perpendicular to Line 2. What is the slope of Line 3?

6. Sketch the graph of each line.

   (a) $y = -\frac{1}{2}x + 3$ (b) $y = \frac{4}{5}x - 1$ (c) $3x - y = 4$

   (d) $x + y = 2$ (e) $3x + 5y = 10$

7. (a) Sketch the graph of $x = -2$. (b) Sketch the graph of $y = 4$.

8. Find the equation of the line through the two points **algebraically**.

   (a) $(3, 6),\ (12, 18)$ (b) $(-3, 9),\ (6, 3)$ (c) $(3, 2),\ (-5, 2)$

   (d) $(-8, -7),\ (-4, -6)$ (e) $(2, -1),\ (6, -4)$ (f) $(-2, 4),\ (1, 4)$

9. The equation $5x - 3y = 2$ is the equation of a line.

   (a) Give the slope of the line.
   (b) What is the slope of any line parallel to the line with the given equation?
   (c) What is the slope of any line perpendicular to the line with the given equation?

10. Are the two lines $3x - 4y = 7$ and $8x - 6y = 1$ perpendicular, parallel, or neither?

11. Find the equation of the line containing the point $P(2, 1)$ and perpendicular to the line with equation $2x + 3y = 9$. Give your answer in slope-intercept form $(y = mx + b)$.

12. Find the equation of a line with x-intercept -4 and y-intercept 3.

13. Find the equation of a horizontal line through the point $(2, -5)$.

14. Find the equation of the line through the point $(2, -3)$ and parallel to the line with equation $4x + 5y = 15$.

15. Find the equation of the line that contains the point $(3, 4)$ and is perpendicular to the line with equation $3x - 2y = 3$.

16. Find the equation of the line that is parallel to $y = -5x + 7$ and has x-intercept $x = 2$.

17. Find the equation of the line that is perpendicular to the line with equation $3x + 4y = -1$ and contains the point $(3, 2)$.

18. Determine, *without graphing*, whether the points $A(-2, 0)$, $B(2, 3)$ and $C(-6, -3)$ lie on the same line. Show work supporting your answer. (**Hint:** Find the slope of the line through points A and B, then through B and C. How should they compare if the points are all on the same line?)

19. Find the equation of the line through $(-5, 2)$ that is parallel to the line through $(-3, -1)$ and $(2, 6)$.

1.6 Systems of Two Linear Equations

Performance Criteria:

1. (j) Solve a system of two linear equations in two unknowns by the addition method; solve a system of two linear equations in two unknowns by the substitution method.

   (k) Solve a problem using a system of two linear equations in two unknowns.

The Addition Method

Consider the two equations

$$\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$$

Taken together, we call them a **system of equations**. In particular, this is a system of *linear* equations. From past experience it should be clear that each equation has infinitely many solution pairs. When faced with a system of linear equations, our goal will be to *solve the system*. This means find a value of x and a value of y which, when taken together, make *BOTH* equations true. In this case you can easily verify that the values $x = 5,\ y = -2$ make both equations true. We say that the ordered pair $(5, -2)$ is a solution to the system of equations; it turns out it is the *only* solution.

You might recall that there are two methods for solving systems of linear equations, the **addition method** and the **substitution method**. Here is a simple example of the addition method:

◇ **Example 1.6(a):** Solve the system $\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned}$ using the addition method.

Solution: The idea is to be able to add the two equations and get either the x terms or the y terms to go away. If we multiply the first equation by 2 and add it to the second equation the x terms will go away. We'll then solve for y:

$$\begin{array}{rcr} x - 3y = 6 & \overset{\text{times } 2}{\Longrightarrow} & 2x - 6y = 12 \\ -2x + 5y = -5 & \Longrightarrow & \underline{-2x + 5y = -5} \\ & & -y = 7 \\ & & y = -7 \end{array}$$

We then substitute this value of y into either of the original equations to get y:

$$\begin{aligned} x - 3(-7) &= 6 \\ x + 21 &= 6 \\ x &= -15 \end{aligned}$$

The solution to the system is then $x = -15,\ y = -7,$ or $(-15, -7)$.

In many cases we must multiply each of the original equations by a different value in order to eliminate one of the unknonwns when adding. This is demonstrated in the next example.

◇ **Example 1.6(b):** Solve the system $\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$ using the addition method.

Solution: Here if we multiply the first equation by 5 and the second equation by 4 the y terms will go away when we add them together. We'll then solve for x:

$$\begin{aligned} 2x - 4y = 18 &\overset{\text{times } 5}{\Longrightarrow} & 10x - 20y &= 90 \\ 3x + 5y = \ \ 5 &\overset{\text{times } 4}{\Longrightarrow} & \underline{12x + 20y = 20} \\ & & 22x &= 110 \\ & & x &= 5 \end{aligned}$$

We then substitute this value of x into any one of our equations to get y:

$$\begin{aligned} 3(5) + 5y &= \ \ 5 \\ 15 + 5y &= \ \ 5 \\ 5y &= -10 \\ y &= -2 \end{aligned}$$

The solution to the system is then $x = 5$, $y = -2$, or $(5, -2)$. This can easily be verified by substituting into the two original equations:

$$\begin{aligned} 2(5) - 4(-2) &= 10 + 8 = 18 \\ 3(5) + 5(-2) &= 15 - 10 = 5 \end{aligned}$$

These steps are summarized here:

The Addition Method

To solve a system of two linear equations by the addition method,

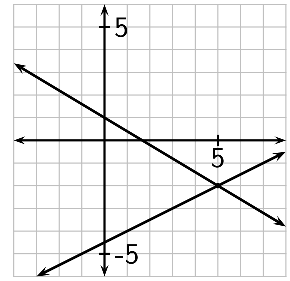
1) Multiply each equation by something as needed in order to make the coefficients of either x or y the same but opposite in sign. *Both equations will not be multiplied by the same number!*

2) Add the two equations and solve the resulting equation for whichever unknown remains.

3) Substitute that value into either equation and solve for the other unknown.

The following example shows the graphical interpretation of what is going on when we solve a system of two linear equations in two unknowns.

◇ **Example 1.6(c):** Solve each of the equations from Example 1.6(b) for y. Then graph the two lines on the same grid and determine where they cross.



Solution: Solving the two equations for y we get

$$y = \frac{1}{2}x - 4\frac{1}{2} \qquad \text{and} \qquad y = -\frac{3}{5}x + 1.$$

Both lines are graphed on the grid to the right. Note that they cross at $(5, -2)$, the solution to the system of equations.

The Substitution Method

We will now describe the substitution method which works best when one of x or y has coefficient 1 or -1 in one of the equations, then give an example of how it works.

> **The Substitution Method**
>
> To solve a system of two linear equations by the substitution method,
>
> 1) Pick the equation in which the coefficient of one of the unknowns is either one or negative one. Solve that equation for that unknown.
>
> 2) Substitute the expression for that unknown into *the other* equation and solve for the unknown.
>
> 3) Substitute that value into any equation and solve for the other unknown. *It is generally easiest to substitute it into the solved equation from step one.*

◇ **Example 1.6(d):** Solve the system of equations $\begin{array}{rcl} x - 3y & = & 6 \\ -2x + 5y & = & -5 \end{array}$ using the substitution method.

Solution: Solve the first equation for x: $x = 3y+6$. Substitute into the second equation and solve for y :

$$\begin{aligned} -2(3y+6) + 5y &= -5 \\ -6y - 12 + 5y &= -5 \\ -y &= 7 \\ y &= -7 \end{aligned}$$

Substitute $y = -7$ into either original equation or $x = 3y + 6$ to find x:

$$x = 3(-7) + 6 = -21 + 6 = -15$$

The solution to the system is $(-15, -7)$.

Applications

Systems of two linear equations can be quite handy for solving problems, as seen in the next example.

◇ **Example 1.6(e):** The length of a rectangle is 3.2 inches less than twice its width, and its perimeter is 51.6 inches. Find the length and width of the rectangle.

Solution: Let l represent the length of the rectangle and let w represent the width. Then we have $l = 2w - 3.2$ and $2l + 2w = 51.6$. This is all set for substitution:

$$\begin{aligned} 2(2w - 3.2) + 2w &= 51.6 \\ 6w - 6.4 &= 51.6 \\ 6w &= 63.5 \\ w &= 10.6 \end{aligned}$$

Substituting this value into $l = 2w - 3.2$ we get $l = 2(10.6) - 3.2 = 18$, so the length is 18 inches, and the width is 10.6 inches.

When "Things Go Wrong"

Consider the the following two systems of equations:

$$\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned} \qquad\qquad \begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= -6 \end{aligned}$$

If we attempt to solve the first system by the addition method, multiplying the first equation by two and then adding, we get $0 = 7$. This is obviously an erroneous statement! It is trying to tell us that the system has no solution. Doing the same for the second system results in $0 = 0$, which is true, but not helpful. In this case the system has infinitely many solutions; any (x, y) pair that is a solution to the first equation is also a solution to the second equation. Both of these situations are of interest in certain applications, but that goes beyond the scope of this course.

Section 1.6 Exercises

To Solutions

1. Solve each of the following systems by both the addition method and the substitution method.

   (a) $\begin{aligned} 2x + y &= 13 \\ -5x + 3y &= 6 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (c) $\begin{aligned} x + y &= 3 \\ 2x + 3y &= -4 \end{aligned}$

2. Solve each of the following systems by the addition method.

   (a) $\begin{aligned} 7x - 6y &= 13 \\ 6x - 5y &= 11 \end{aligned}$ (b) $\begin{aligned} 5x + 3y &= 7 \\ 3x - 5y &= -23 \end{aligned}$ (c) $\begin{aligned} 5x - 3y &= -11 \\ 7x + 6y &= -12 \end{aligned}$

3. Do the following for each of the systems below.

   (i) Solve each equation for y; do not use decimals!

   (ii) Plot both lines using a graphing calculator or online grapher. Give the solution to the system, based on what you see.

   (iii) Solve the system by either the addition method or the substitution method. If your solution does not agree with what you got in part (b), check your work.

   (a) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -7 \\ -2x + 5y &= 9 \end{aligned}$ (c) $\begin{aligned} 4x + y &= 14 \\ 2x + 3y &= 12 \end{aligned}$

4. Consider the system $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned}$.

   (a) Solve the system by either the addition method or the substitution method.

   (b) Solve each equation for y; again, do not use decimals!

   (c) Plot both lines using a graphing calculator or online grapher.

   (d) Discuss the solution to the system,and how it is illustrated by the graph.

5. Repeat the steps of Exercise 6 for the system $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned}$.

6. Consider the system of equations $\begin{aligned} 2x - 3y &= 4 \\ 4x + 5y &= 3 \end{aligned}$.

   (a) Solve for x by using the addition method to eliminate y. Your answer should be a fraction.

   (b) Ordinarily you would substitute your answer to (a) into either equation to find the other unknown. However, dealing with the fraction that you got for part (a) could be difficult and annoying. Instead, use the addition method again, but eliminate x to find y.

7. Consider the system of equations $\begin{aligned} \frac{1}{2}x - \frac{1}{3}y &= 2 \\ \frac{1}{4}x + \frac{2}{3}y &= 6 \end{aligned}$. The steps below indicate how to solve such a system of equations.

   (a) Multiply both sides of the first equation by the least common denominator to "kill off" all fractions.

   (b) Repeat for the second equation.

   (c) You now have a new system of equations without fractional coefficients. Solve that system by the addition method.

8. An airplane flying with the wind travels 1200 miles in 2 hours. The return trip, against the wind, takes $2\frac{1}{2}$ hours. Assuming that the speed of both the wind and the plane were constant during both trips, what was the speed of the wind, and what was the speed of the plane?

9. You are in line at a ticket window. There are two more people ahead of you than there are behind you. The number of people in the entire line (don't forget to count yourself!) is three times the number of people behind you. How many people are ahead of you in line, and how many are behind you?

10. An economist models the market for wheat by the following equations:

Supply Equation	$y = 8.33p - 14.58$
Demand Equation	$y = -1.39p + 23.35$

Here p is the price per bushel (in dollars) and y is the number of bushels produced and sold (in **millions** of bushels). Find the number of bushels produced and sold at equilibrium. (Equilibrium refers to the situation in which supply and demand are equal.)

1.7 More on Graphing Equations

Performance Criteria:

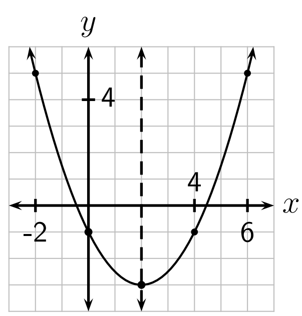
1. (l) Graph $y = ax^2 + bx + c$ by finding and plotting the vertex, y-intercept, and several other points.

   (m) Identify or create the graphs of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

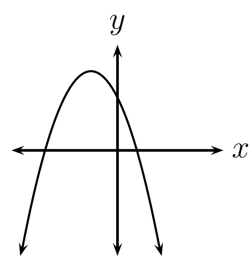
   (n) Graph variations of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

In Section 1.4 we introduced the idea of showing solutions to equations in two unknowns as graphs, and in Section 1.5 we saw how to obtain the graph of $Ax+By = C$ or $y = mx+b$ as efficiently as possible. In this section we will refine our procedure for graphing other equations.

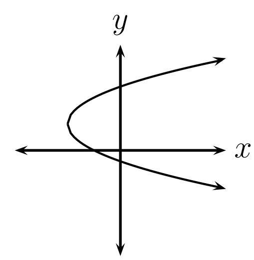
The shapes shown in the graphs below are all **parabolas**. We say that the first two open upward and downward, respectively, and the second two open to the right and left, respectively. Note that we could write $y = mx + b$ as $y = bx + c$; the letters we use for the slope and y-intercept can be any we choose. If we add ax^2 (for a not zero) to the right side of the second of these we get the standard equation $y = ax^2 + bx + c$ of a parabola. When $a > 0$ the graph is an upward opening parabola, and when $a < 0$ the graph is parabola that opens downward. These situations are shown in the first two graphs below. When the equation is $x = ay^2 + by + c$ we get graphs like the two to the right below, the first of them when $a > 0$ and the second one when $a < 0$. We will usually work with equations of the form $y = ax^2 + bx + c$ (where any of a, b or c might be negative and either or both of of b and c can be zero).



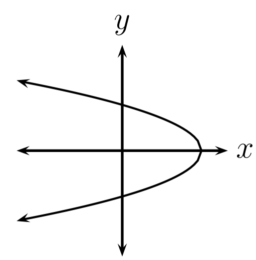
$y = ax^2 + bx + c, \quad a > 0$



$y = ax^2 + bx + c$
$a < 0$



$x = ay^2 + by + c$
$a < 0$



$x = ay^2 + by + c$
$a < 0$

Looking at the first parabola above, the point $(2, -3)$ is called the **vertex** of the parabola. It is the first point we should plot when graphing a parabola and, for parabolas opening up or down, it is important in applications because it is the lowest or highest point on the graph. The x-coordinate of the vertex of a parabola with equation $y = ax^2 + bx + c$ is given by $x = -\dfrac{b}{2a}$, and the y-coordinate of the vertex is found by substituting that x value into the equation.

◇ **Example 1.7(a):** Determine the coordinates of the vertex of the parabola with equation $y = -\frac{1}{2}x^2 + 2x - 1$, and whether the parabola opens up or down.

Solution: The x-coordinate of the vertex is $x = -\dfrac{2}{2(-\frac{1}{2})} = -\dfrac{2}{-1} = 2$. The y-coordinate of the vertex is found by substituting the value 2 that we just found for x into the original equation:

$$y = -\frac{1}{2}(2)^2 + 2(2) - 1 = -2 + 4 - 1 = 1$$

Therefore the vertex of the parabola is $(2, 1)$ and, because $a = -\frac{1}{2} < 0$, it opens down.

Another key idea for graphing parabolas can be seen in the graph of the first parabola on the previous page: The graph is *symmetric about the line with equation* $x = 2$. That line is called the **axis of symmetry** for the parabola. When graphing, the importance of this is that every point on the parabola (besides the vertex) has a "partner" on the other side of the parabola, an equal distance from the axis of symmetry as the original point. In the first graph on the previous page we can see that the point $(4, -1)$ has $(0, -1)$ as its partner, and the points $(-2, 5)$ and $(6, 5)$ pair up in the same way.

The observations just made lead us to the following method for efficiently graphing equations of the form $y = ax^2 + bx + c$:

Graphing $y = ax^2 + bx + c$

1) Recognize from the form of the equation that the graph will be a parabola, and determine whether the parabola opens upward or downward.

2) Use $x = -\dfrac{b}{2a}$ to find the x-coordinate of the vertex, then evaluate the equation for that value of x to find the y-coordinate of the vertex. Plot that point.

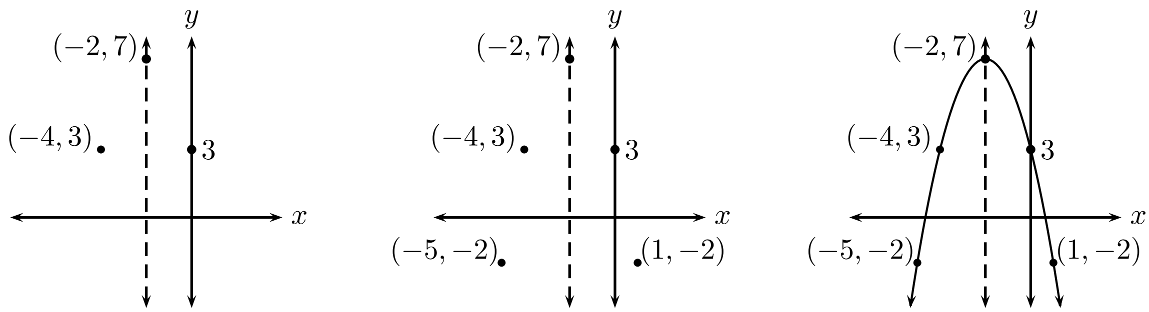
3) Find and plot the y-intercept and the point opposite it. Sketch in the line of symmetry if you need to in order to do this.

4) Pick an x value other than ones used for the three points that you have plotted so far and use it to evaluate the equation. Plot the resulting point and the point opposite it.

4) Draw the graph of the equation. As always, include arrowheads to show where the the graph continues beyond the edges of the graph.

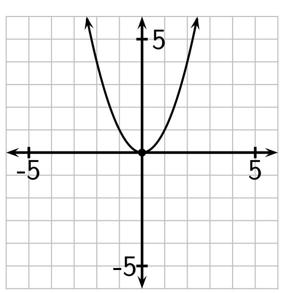
◇ **Example 1.7(b):** Graph the equation $y = -x^2 - 4x + 3$. Indicate clearly and accurately five points on the parabola.

Solution: First we note that the graph will be a parabola opening downward, with y-intercept 3. (This is obtained, *as always*, by letting $x = 0$.) The x-coordinate of the vertex is $x = -\dfrac{-4}{2(-1)} = -2$. Evaluating the equation for this value of x gives us $y = -(-2)^2 - 4(-2) + 5 = -4 + 8 + 3 = 7$, so the vertex is $(-2, 7)$. On the first graph at the top of the next page we plot the vertex, y-intercept, the axis of symmetry, and the point opposite the y-intercept, $(-4, 3)$. To get two more points we can evaluate the equation for $x = 1$ to get $y = -1^2 - 4(1) + 3 = -2$. We then plot the resulting point $(1, -2)$ and the point opposite it, as done on the second graph. The third graph then shows the graph of the function with our five points clearly and accurately plotted.

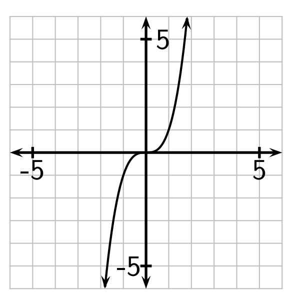


Some Basic Functions

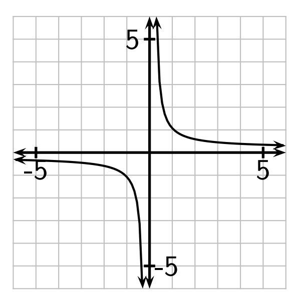
In Section 1.5 we found how to quickly and easily graph any line, and we just found out how to efficiently graph parabolas. We now consider the graphs of some other equations of interest. First let's recall the graphs, that we constructed in the exercises of Section 1.4, of some basic equations:



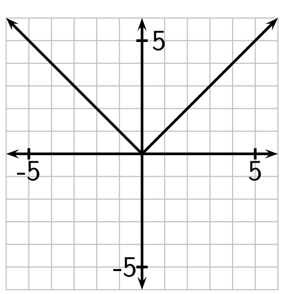
$y = x^2$



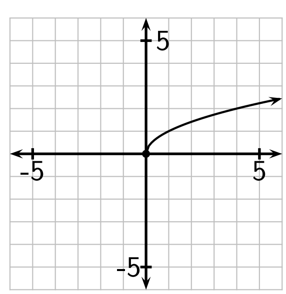
$y = x^3$



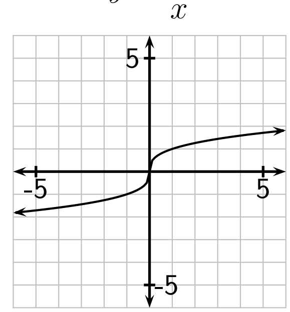
$y = \dfrac{1}{x}$



$y = |x|$



$y = \sqrt{x}$



$y = \sqrt[3]{x}$

The above graphs illustrate a few things to keep in mind, beyond what we already know about the graphs of $y = mx + b$ and $y = ax^2 + bx + c$:

- Graphs of equations containing absolute values are usually vee-shaped.
- Graphs of equations containing square roots are usually half parabolas.
- Graphs of equations whose denominator is zero for one or more values of x will have two or more separate pieces.

We now introduce an important and useful principle, which we will relate to something that we already know. Suppose that we are trying to solve the equation $x^2 = 2x + 3$. We get zero on one side and factor to get $(x - 3)(x + 1) = 0$, for which we can see that $x = 3$ and $x = -1$ are solutions. In general, when we see an expression like $x+a$, $x-a$, or $a-x$ for some number a, the value that makes the expression zero is important for some reason. Consider the equations

$$y = |x - 3|, \qquad y = \sqrt{x+4}, \qquad \text{and} \qquad y = \frac{6}{2-x}$$

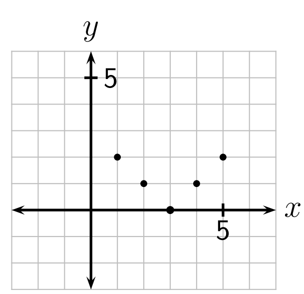
For the first of these, $x = 3$ is an important value for a reason you will soon see. For the second equation $x = -4$ is important because if x is any number less than -4, y will be undefined, but for $x = -4$ and larger, y *IS* defined. y will be undefined for $x = 2$ in the last equation, but y values *CAN* be obtained for *ANY* other value of x. Here is how to use all of this information:

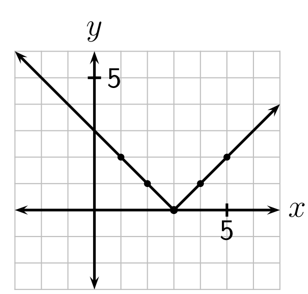
> When graphing an equation containing one of the expressions $x + a$, $x - a$, or $a-x$, first determine the value of x that makes the expression zero, which we will call a **critical value**. If possible, find the value of y for the critical value of x. Then find additional values of y using x values obtained by "working outward" from from the critical value, as illustrated in the following examples.

◇ **Example 1.7(c):** Graph the equation $y = |x - 3|$.

Solution: Because the equation contains an absolute value we expect the graph to be a vee, and $x = 3$ is a critical value because it makes $x - 3$ zero. We find the values of y for $x = 3$ and several values on either side of three, as shown in the table below and to the left. The points are plotted on the left grid, where the vee becomes apparent, and the final graph is shown to the right below.

x	y
1	2
2	1
3	0
4	1
5	2

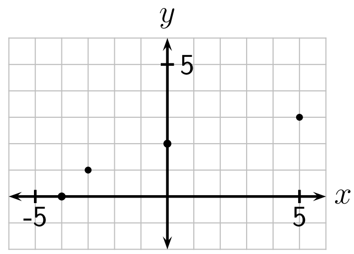


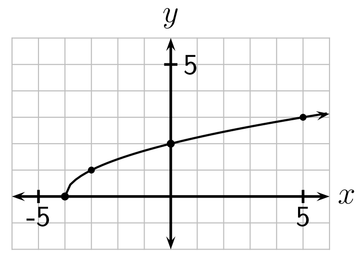


◇ **Example 1.7(d):** Graph the equation $y = \sqrt{x+4}$.

Solution: Here we expect the graph to be a half parabola, and we also see that $(-4)+4 = 0$, so $x = -4$ is a critical value. y is not defined for any number less than negative four, so we want to use -4 and larger for x. We should also choose values of x that make $x+4$ a perfect square like 0, 1, 4, 9, 16,... This is done in the table to the left at the top of the next page, those points are plotted on the middle grid, and the final graph is on the right grid. Note that the graph ends at $(-4, 0)$, as indicated by the solid dot there, but continues in the other direction, as shown by the arrowhead.

x	y
-4	0
-3	1
0	2
5	3

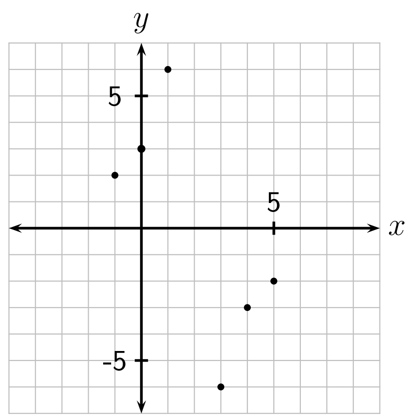


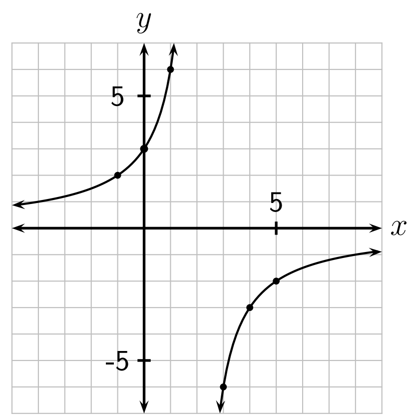


◇ **Example 1.7(e):** Graph the equation $y = \dfrac{6}{2-x}$.

Solution: Here we expect the graph to be a half parabola, and we also see that $(-4)+4 = 0$, so $x = -4$ is a critical value. y is not defined for any number less than negative four,so we want to use -4 and larger for x. We should also choose values of x that make $x+4$ a perfect square like 0, 1, 4, 9, 16,... This is done in the table to the left at the top of the next page, those points are plotted on the left grid below, and the final graph is on the right grid below. Note that the graph ends at $(-4, 0)$, as indicated by the solid dot there, but continues in the other direction, as shown by the arrowhead.

x	y
-1	2
0	3
1	6
3	-6
4	-3
5	-2
8	-1





Section 1.7 Exercises

To Solutions

1. For each of the following quadratic equations, first determine whether its graph will be a parabola opening upward or opening downward. Then find the coordinates of the vertex.

   (a) $y = -3x^2 - 6x + 2$ (b) $y = x^2 - 6x + 5$ (c) $y = \frac{1}{2}x^2 + 2x + 5$

2. Graph the solution set of each of the equations from Exercise 1.

3. Graph $y = x^2 + 2x + 3$, indicating five points on the graph clearly and accurately.

4. Graph the equation $y = x^2 + 4x - 5$. Indicate clearly and accurately five points on the parabola. How does the result compare with that from Example 1.7(b), in terms of the direction it opens, its intercepts, and its vertex??

5. Discuss the x-intercept situation for equations of the form $y = ax^2 + bx + c$. Do they always have at least one x-intercept? Can they have more than one? Sketch some parabolas that are arranged in different ways (but always opening up or down) relative to the coordinate axes to help you with this.

6. (a) A parabola has x-intercepts 3 and 10. What is the x-coordinate of the vertex?
   (b) The vertex of a *different* parabola is $(3, 2)$, and the point $(5, 4)$ is on the parabola. Give another point on the parabola.
   (c) The vertex of yet another parabola is $(-2, 9)$. The y-intercept of the parabola is 5 and one x-intercept is -5. Give the other x-intercept and one other point on the parabola.

7. Consider the equation $y = x^2 - 6x - 7$. Here you will graph the equation in a slightly different manner than used in Example 1.7(b).

   (a) You know the graph will be a parabola - which way does it open?
   (b) Find the y-intercept and plot it.
   (c) Find the x-intercepts, and plot them.
   (d) Sketch the line of symmetry on your graph - you should be able to tell where it is, based on the x-intercepts. What is the x-coordinate of the vertex?
   (e) Find the y-coordinate of the vertex of the parabola by substituting the x-coordinate into the equation.
   (f) Plot the vertex and the point opposite the y-intercept and graph the equation.
   (g) Where (at what x value) does the minimum or maximum value of y occur? Is y a minimum, or a maximum, there? What is the minimum or maximum value of y?

8. Here you will find that the method used in the previous exercise for finding the x-coordinate of the vertex of a parabola will not always work, but that you can always find the x-coordinate of the vertex using the formula $x = -\frac{b}{2a}$.

   (a) Find the x-intercepts of $y = x^2 - 6x + 13$. What goes wrong here?

   (b) Which way does the parabola open?

   (c) Find the vertex or the parabola.

   (d) Tell how your answers to (b) and (c) explain what happened in (a).

   (e) Give the coordinates of four other points on the parabola.

9. Consider the equation $y = 10 + 3x - x^2$.

   (a) What are the values of a, b and c for this parabola?

   (b) Use the procedure of Example 1.7(b) to sketch the graph of the parabola.

10. Some equations are given below. For each, determine the most critical value of x for graphing the equation, then either find the corresponding y value or state that the equation is undefined for the critical x value. (In one case there is no obvious critical x value.)

   (a) $y = \dfrac{4}{x+3}$ (b) $y = x^2 - 5x + 6$ (c) $y = \dfrac{6}{2-x}$

   (d) $y = \sqrt{x+5}$ (e) $y = -\dfrac{2}{3}x + 4$ (f) $y = \sqrt{4-x}$

   (g) $y = \dfrac{6}{x-1}$ (h) $y = \dfrac{8}{x^2}$ (i) $y = \sqrt{x+3} - 1$

11. Graph each of the equations from Exercise 10, indicating clearly at least four points *with integer coordinates* on each graph.

1.8 Chapter 1 Exercises

To Solutions

1. Solve each equation. When the quadratic equation is required, give both exact solutions and approximate (decimal) solutions rounded to the nearest tenth.

   (a) $2(x+4)-5(x+10)=6$

   (b) $\dfrac{8}{x^2-9}+\dfrac{4}{x+3}=\dfrac{2}{x-3}$

   (c) $2|x-5|-3=5$

   (d) $x^2-6x+7=0$

   (e) $15x^2=20x$

   (f) $\sqrt{-2x+1}=-3$

   (g) $5x+1=2x+8$

   (h) $2y-8=-y^2$

   (i) $\dfrac{3}{2}+\dfrac{5}{x-3}=\dfrac{x+9}{2x-6}$

   (j) $36-x^2=0$

2. A student is solving the equation $5-2(x+4)=x-2$ and obtains the solution $x=5$.

   (a) Convince them that their solution is incorrect, *without solving the equation*.

   (b) Can you guess what their error was? Show how they obtained their solution, and tell where they went wrong.

   (c) For the same equation, another student got the incorrect solution $x=-7$. Show how they obtained their solution, and tell where they went wrong.

   (d) Solve the equation and verify that your solution is correct.

3. (a) Create an equation of the form $|x+a|=b$ with solutions $x=2,8$.

   (b) Create an equation of the same form with solutions $x=3,11$.

   (c) Suppose that we want an equation of the same form to have solutions x_1 and x_2. Find formulas for obtaining the values of a and b from x_1 and x_2.

   (d) Check your answer to (c) by finding an equation of the given form with solutions $x=-5,1$. Check to make sure that those values really are solutions!

4. The length of a rectangle is 3.2 inches less than twice its width, and its perimeter is 57.2 inches. Find the length and width of the rectangle.

5. The width of a rectangle is three more than half the length. The perimeter is 39. How long are the sides of the rectangle?

6. The cost of a compact disc, with 6% sales tax, was $10.55. What was the price of the compact disc?

7. Solve each equation for y. Give your answers in $y=mx+b$ form.

   (a) $5x+2y=20$

   (b) $2x-3y=6$

   (c) $3x+5y+10=0$

8. Solve each equation for x.

   (a) $ax+bx=c$

   (b) $ax+3=cx-7$

9. A projectile is shot upward from ground level with an initial velocity of 48 feet per second.

   (a) When is the projectile at a height of 20 feet?

   (b) When does the projectile hit the ground?

   (c) What is the height of the projectile at 1.92 seconds? **Round your answer to the nearest tenth of a foot.**

   (d) When is the projectile at a height of 32 feet?

   (e) When is the projectile at a height of 36 feet? Your answer here is a bit different than your answers to (a) and (d). What is happening physically with the projectile?

   (f) When is the projectile at a height of 40 feet?

10. A manufacturer of small calculators knows that the number x of calculators that it can sell each week is related to the price per calculator p by the equation $x = 1300 - 100p$. The weekly revenue (money they bring in) is the price times the number of calculators: $R = px = p(1300 - 100p)$. What price should they set for the cartridges if they want the weekly revenue to be $4225?

11. Find the intercepts for each of the following equations algebraically.

    (a) $y = x^2 - 2x$ (b) $x + 2y = 4$ (c) $x - y^2 = 1$

    (d) $x^2 + y^2 = 25$ (e) $4x - 5y = 20$ (f) $x = \sqrt{y+4}$

    (g) $y = 2|x| - 3$

12. Find the equation of the line through the two points **algebraically**.

    (a) $(-4,-2),\ (-2,4)$ (b) $(1,7),\ (1,-1)$ (c) $(-1,-5),\ (1,3)$

    (d) $(1,7),(3,11)$ (e) $(-6,-2),\ (5,-3)$ (f) $(-2,-5),\ (2,5)$

13. Consider the system of three equations in three unknowns x, y and z:

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned}$$

    The steps below indicate how to solve such a system of equations - it is essentially the same as solving a system of two equations in two unknowns.

    (a) Use the addition method with the first two equations to eliminate x.

    (b) Use the addition method with the first and third equations to eliminate x.

    (c) Your answers to (a) and (b) are a new system of two equations with two unknowns. Use the addition method to eliminate y and solve for z.

    (d) Substitute the value you found for z into one of the equations containing y and z to find y.

(e) You should now know y and z. Substitute them into ANY of the three equations to find x.

(f) You now have what we call an *ordered triple* (x, y, z). Check your solution by substituting those three numbers into each of the original equations to make sure that they make all three equations true.

14. Solve each of the following systems of equations.

(a) $$\begin{aligned} x + 2y - z &= -1 \\ 2x - y + 3z &= 13 \\ 3x - 2y &= 6 \end{aligned}$$

(b) $$\begin{aligned} 2x - y + z &= 6 \\ 4x + 3y - z &= 1 \\ -4x - 8y + 2z &= 1 \end{aligned}$$

(c) $$\begin{aligned} x - 2y + 3z &= 4 \\ 2x + y - 4z &= 3 \\ -3x + 4y - z &= -2 \end{aligned}$$

(d) $$\begin{aligned} x + 3y - z &= -3 \\ 3x - y + 2z &= 1 \\ 2x - y + z &= -1 \end{aligned}$$

15. The graph of $y = 10 + 3x - x^2$ is a parabola. Determine whether it opens up or down, and give the coordinates of its vertex.

16. For each of the quadratic functions below,

- give the general form of the graph (parabola opening upward, parabola opening downward),
- give the coordinates of the vertex,
- give the y-intercept of the graph and the coordinates of one other point on the graph,
- sketch the graph of the function,
- tell what the maximum or minimum value of the function is (stating whether it is a maximum or a minimum) and where it occurs.

(a) $f(x) = -x^2 + 2x + 15$

(b) $g(x) = x^2 - 9x + 14$

(c) $h(x) = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{9}{4}$

(d) $s(t) = \frac{1}{2}t^2 + 3t + \frac{9}{2}$

17. Consider the equation $y = (x - 3)^2 - 4$.

(a) Get the equation in the form $f(x) = ax^2 + bx + c$.

(b) Find the coordinates of the vertex of the parabola. Do you see any way that they might relate to the original form in which the function was given?

(c) Find the x and y-intercepts of the equation and sketch its graph.

(d) On the same coordinate grid, sketch the graph of $y = x^2$ *accurately*. (Find some ordered pairs that satisfy the equation and plot them to help with this.) Compare the two graphs. Check your graphs with your calculator.

18. Consider the equation $y = -\frac{1}{2}(x-2)(x+3)$. Do all of the following without multiplying this out!

   (a) If you were to multiply it out, what would the value of a be? Which way does the parabola open?

   (b) Find the y-intercept of the equation.

   (c) Find the x-intercepts of the equation. In this case it should be easier to find the x-intercepts than the y-intercept!

   (d) Find the coordinates of the vertex.

   (e) Graph the equation, labelling the vertex with its coordinates.

19. Graph $h(x) = \sqrt{9 - x^2}$. Check your answer using a graphing utility.

2 Introduction to Functions

Outcome/Performance Criteria:

2. Understand general concepts relating to functions and their graphs.

   (a) Evaluate a function, and find all values for which a function takes a particular value, from the equation of the function.

   (b) Graph a function; evaluate a function, and find all values for which a function takes a particular value, from the graph of the function.

   (c) Describe a set using set builder notation or interval notation.

   (d) Give the domain and range of a function from the equation of the function.

   (e) Give the domain and range of a function from its graph.

   (f) Given the graph of a function,

   - give the intervals on which the function is increasing (decreasing)
   - give the intervals on which the function is positive (negative)
   - give the maxima and minima of the function and their locations, and determine whether each is relative or absolute
   - identify the function as even, odd, or neither

   (g) Given the definitions of even and odd functions, determine whether a function is even, odd or neither from its equation.

   (h) Use a linear model to solve problems; create a linear model for a given situation.

   (i) Interpret the slope and intercept of a linear model.

   (j) Solve a problem using a function whose equation is given.

   (k) Determine the equation of a function modeling a situation.

   (l) Give the feasible domain of a function modeling a situation.

   (m) Given the graph of a function, sketch or identify translations of the function.

   (n) Given the graph of a function, sketch or identify vertical reflections of the function.

2.1 Functions and Their Graphs

Performance Criteria:

2. (a) Evaluate a function, and find all values for which a function takes a particular value, from the equation of the function.
   (b) Graph a function; evaluate a function, and find all values for which a function takes a particular value, from the graph of the function.

Introduction

Consider the three equations

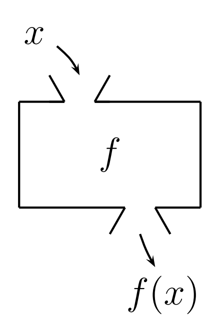
$$y = 3x - 8 \qquad\qquad y = \sqrt{25 - x^2} \qquad\qquad y = 2x^2 + x - 5$$

For each equation, if we choose a value for x and substitute it into the equation, we can find a corresponding value for y, and *there is no doubt what that value of y is.* (In the second equation we have to be a bit careful about what value of x we choose, but we'll get into that later!) Whenever we have an equation for which any allowable value of x leads to only one value of y we say that "y is a function of x."

Of course the choice of letters doesn't really matter. If we were to rewrite the last equation as $v = 2w^2 + w - 5$ we would say that v is a function of w, but it is really the same function because v is obtained from w in the same way that y is obtained from x. Functions are a very fundamental concept of math and science, and we will spend most of this term studying them and their applications. Many of you will then take trigonometry after this course, and most of that course is devoted to studying special kinds of functions called **trigonometric functions**.

Rather than saying "y is a function of x" each time, when dealing with such an equation we will just say it is a function. Notice that finding y in these three functions amounts to simply substituting a value for x and doing a little arithmetic. Each of these functions can be thought of as a "machine" that we simply feed an x value into and get out a corresponding value for y. There is an alternative notation for functions that is very efficient once a person is used to it. Suppose that we name the function $y = 3x - 8$ with the letter f; we write this as $f(x) = 3x - 8$. We then think of f as a sort of "black box" that we feed numbers into and get numbers out of. We denote by $f(-2)$ the value that comes out when -2 is fed into the machine, so $f(-2) = -14$.

In general, the "input" of a function f is x, and the output is $f(x)$. The picture to the right illustrates this idea of a function as a "machine." If we put a number like 5 into the machine, the symbolism $f(5)$ represents the "output," the number that will come out when 5 is put in. *It is important to keep in mind that the symbolism $f(x)$ really stands for one number.* Technically speaking, a function is simply a way to associate to any "input" value a corresponding "output" value. We will always want to describe how to get the output of a function for any given input, and the two main ways we will do this in this course are by



i) giving an equation describing how the output is obtained from the input, as we have been doing so far.

ii) giving a graph (picture) of how the outputs relate to the inputs.

Let's take a look at the first way of describing a function.

Equations of Functions

Consider again the equation $y = 3x - 8$. We think of x as the input for this equation, and y is the output. Using the function notation described on the previous page, we can replace y with $f(x)$ to get $f(x) = 3x - 8$. Our first two concerns are these:

- Given an input for a function, what is the corresponding output?
- Given an output for a function what are the corresponding inputs?

You will note the way these two are written. Given an input for a function, there will be at most one output - this is what makes a function different from some of the other equations we will work with. However, *one output can come from more than one input.* A simple example of this is the function $y = x^2$; the output 25 could occur when the input is either of 5 or -5.

There are very specific ways that you will be asked the above two questions, and it is important that you get used to them and can quickly and easily know what you are being asked to do. Let's continue looking at the function $f(x) = 3x - 8$, and suppose you are asked to find $f(-2)$. Recalling that $f(x)$ represents the output for an input of x, in this case you are being asked to find the output when the input is -2. Since the number -2 is in the place of x in the expression $f(x)$, we need to replace the x on the other side of the equal sign with -2 and compute the result.

◇ **Example 2.1(a):** For the function $f(x) = 3x - 8$, find $f(-2)$ and $f(1.4)$.

Solution: $f(-2) = 3(-2) - 8 = -6 - 8 = -14$, $f(1.4) = 3(1.4) - 8 = 4.2 - 4 = -3.8$

When working with functions, I would like you to make it clear what any result you obtain represents. In the above example we arrive at the values -14 and -3.8, and if a person follows the string of steps backward in either computation they can see that -14 is $f(-2)$ and -3.8 is $f(1.4)$.

The second question, "Given an output for a function what are the corresponding inputs?," will be asked like this:

◇ **Example 2.1(b):** For the function $f(x) = 3x - 8$, find all values of x such that $f(x) = 5$.

Solution: There are two clues here as to what we are supposed to be doing. The first is the words "find all values of x," which says that we are supposed to be finding input(s) x. The second clue is that we are given that $f(x)$, which represents an output, is 5. to carry out the task we are asked to do, we replace $f(x)$ with 5 and solve the resulting equation:

$$\begin{aligned} 5 &= 3x - 8 \\ 13 &= 3x \\ \tfrac{13}{3} &= x \end{aligned}$$

Note that the final result is labeled as what we are supposed to find, x.

In this example we were asked to find all values of x for which $f(x) = 5$, and we can verify that our answer is such an x by putting it into the function:

$$f(\tfrac{13}{3}) = 3(\tfrac{13}{3}) - 8 = 13 - 8 = 5$$

◇ **Example 2.1(c):** Let $g(x) = x^2 - 3x$. Find $g(-4)$ and find all values of x such that $g(x) = 10$.

Solution: Finding $g(-4)$ is not difficult, but we need to take a bit of care with signs:

$$g(-4) = (-4)^2 - 3(-4) = 16 + 12 = 28$$

To find all values of x for which $g(x) = 10$, we replace $g(x)$ with 10 in the equation of the function and solve:

$$\begin{aligned} 10 &= x^2 - 3x \\ 0 &= x^2 - 3x - 10 \\ 0 &= (x-5)(x+2) \end{aligned}$$

$$x = 5 \qquad \text{and} \qquad x = -2$$

The process of finding a value of a function for a given input value is called *evaluating* the function. Sometimes you will be asked to evaluate a function for an input that is not a number. This can be a bit confusing, and the algebra is often a bit more involved. Suppose that we have the function $g(x) = x^2 - 3x$, and we are then asked to find $g(x - 2)$. The x in the second of these *IS NOT* the same as the x in the first. Here it is helpful to recognize that we could rewrite the description of the function g as $g(u) = u^2 - 3u$; to find $g(x - 2)$ we then replace all occurrences of u with $x - 2$ and simplify. The following example shows this.

◇ **Example 2.1(d):** Find $g(x - 2)$ for the function $g(x) = x^2 - 3x$.

Solution:

$$\begin{aligned} g(x-2) &= (x-2)^2 - 3(x-2) \\ &= (x^2 - 4x + 4) - 3x + 6 \\ &= x^2 - 7x + 10 \end{aligned}$$

Note where we stop in the above example. We simply put $x-2$ into g, and then simplify and that's it. The end result is the output of a function, and it has an unknown value x in it because the input value $x-2$ had an x in it.

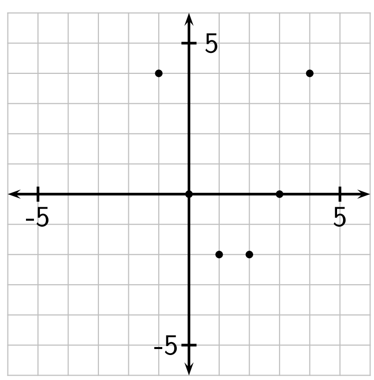
Graphs of Functions

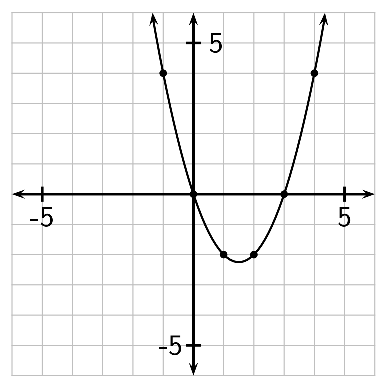
Consider the function $g(x) = x^2 - 3x$. This is simply an equation that lets us find outputs of the function for various inputs. We can graph a function just like any other equation, with the inputs being the x values and the outputs the $g(x)$, or y, values.

◇ **Example 2.1(e):** Graph the function $g(x) = x^2 - 3x$.

Solution: We recognize this as $y = ax^2 + bx + c$, with the output y simply replaced by $g(x)$, so the graph will be a parabola. $a = 1$ and $b = -3$, so the parabola opens upward and the vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2} = 1\frac{1}{2}$. Rather than evaluating the function at $x = -\frac{3}{2}$, which would require working with fractions, we can simply evaluate for integer values working outward from $x = 1\frac{1}{2}$. $g(1) = -2$ and, because $x = 2$ is the same distance from $x = 1\frac{1}{2}$ but on the other side of the x for the vertex, $g(2)$ will also be -2. Similarly, $g(3) = g(0) = 0$ and $g(-1) = g(4) = 4$. The results are in the table to the left below, and those points are plotted in the first graph below. The second graph is the graph of the function $g(x) = x^2 - 3x$. Note the arrowheads indicating the behavior of the graph as it leaves the edges of the grid.

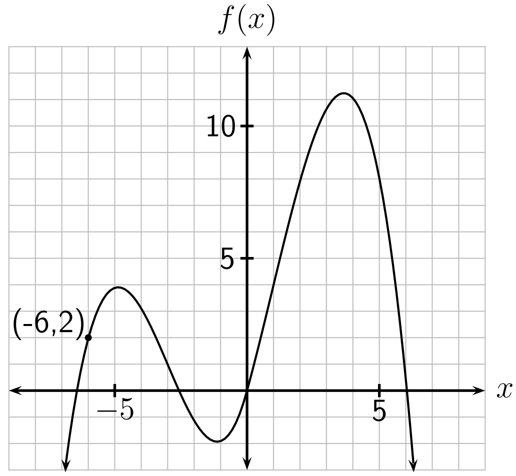
x	$g(x)$
-1	4
0	0
1	-2
2	-2
3	0
4	4

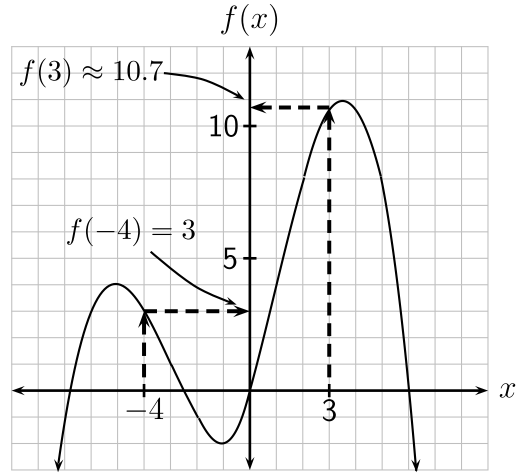




A graph of a function is simply another representation of the function. Just as we found outputs from inputs and inputs from outputs using the equation of a function, we would like to be able to do this from a graph as well. The following example shows how to do this.

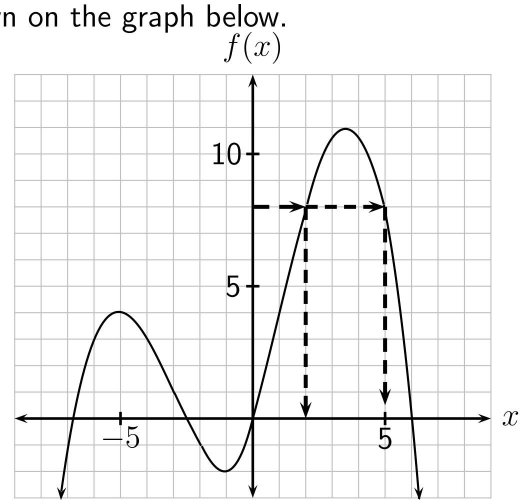
◇ **Example 2.1(f):** Consider the function $y = f(x)$ that has the graph below and to the left. Find $f(-4)$, $f(3)$, and find all values of x such that $f(x) = 8$.





Solution: Note that every point on the graph of the function represents an input-output pair. For example, the pair $(-6, 2)$ indicates that when the input is -6 the output is 2. That is, $f(-6) = 2$. If we want to evaluate the function for a given input value, we simply locate the value on the input (x) axis, go up or down to the graph of the function, then go across to the output (y) axis and read off the output. Using this, $f(-4) = 3$. When working with graphs we often cannot tell exactly what the x or y value corresponding to a particular point is, so we might wish at times to use the approximately equal symbol $\approx$. So we might write $f(3) \approx 10.7$. The way the $f(-4)$ and $f(3)$ are found is shown on the graph above and to the right.

To find all values of x for which $f(x)$ has some value, we "reverse" that process: We locate the given output value on the y-axis, then go across horizontally to all points on the graph with that y value. From each of those we then go straight up or down to find the input (x) value that gives that output. So to find all values of x for which $f(x) = 8$ we go left and right from 8 on the y-axis until we hit points on the graph. From those points we drop down to the x-axis to find the desired x values of 2 and 5. The way these x values are found is shown on the graph below.



Section 2.1 Exercises

To Solutions

1. Let $h(x) = 2x - 7$.

   (a) Find $h(12)$.

   (b) Find all values of x for which $h(x) = 12$.

2. Let $f(x) = 2x^2 - 3x$.

   (a) Find $f(3)$ and $f(-5)$.

   (b) Find all values of x such that $f(x) = 9$.

   (c) Find all values of x such that $f(x) = 1$.

3. Let $g(x) = x - \sqrt{x+3}$.

   (a) Find all values of x such that $g(x) = 3$.

   (b) Find $g(-2)$.

   (c) Find all values of x such that $g(x) = 1.3$, rounding to the hundredth's place.

   (d) List three values of x for which $g(x)$ *CANNOT* be computed, and tell why.

   (e) List three values of x for which $g(x)$ *CAN* be computed.

   (f) With a bit of thought we can determine that $g(x)$ can be computed for all values of x greater than some number, and it cannot be computed for all values of x less than that number. What is the number?

   (g) Can $g(x)$ be found for the number you got for (f)?

4. Let $f(x) = \dfrac{3x+4}{x-2}$.

   (a) Find $f(6)$.

   [(b)] Find all x such that $f(x) = 6$.

   (c) Are there any values for which $g(x)$ cannot be found?

5. Let the function h be given by $h(x) = \sqrt{25 - x^2}$.

   (a) Find $h(-4)$.

   (b) Find $h(2)$, using your calculator. Your answer will be in decimal form and you will have to round it somewhere - round to the hundredth's place.

   (c) Why can't we find $h(7)$?

   (d) For what values of x *CAN* we find $h(x)$? Be sure to consider negative values as well as positive. We will refer to the set of values of x for which we can find $h(x)$ as the **domain** of h. More on this later...

   (e) Find all values of x such that $h(x) = 4$.

6. In this exercise we see a kind of function that is so simple that it can be confusing! Consider the function given by $h(x) = 12$. Find $h(0)$, $h(-43)$, $h(3\frac{4}{7})$. Your answers should all be the same! This is an example of a **constant function**. It is called this because the output has a constant value of 12, regardless of what the input is.

7. Remember that when dealing with a function, the notation $f(x)$ or $g(x)$ is basically just a y value. Keeping this in mind, find the x- and y-intercepts of each of the following functions from previous exercises.

   (a) $h(x) = 2x - 7$
   (b) $f(x) = 2x^2 - 3x$
   (c) $g(x) = x - \sqrt{x+3}$
   (d) $f(x) = \dfrac{3x+4}{x-2}$
   (e) $h(x) = \sqrt{25 - x^2}$
   (f) $h(x) = 12$

8. Let $f(x) = x^2 - 3x$, $g(x) = \sqrt{x+1}$. Find and simplify each of the following.

   (a) $f(x+1)$ (b) $g(2x)$ (c) $f(-2x)$ (d) $g(x-4)$

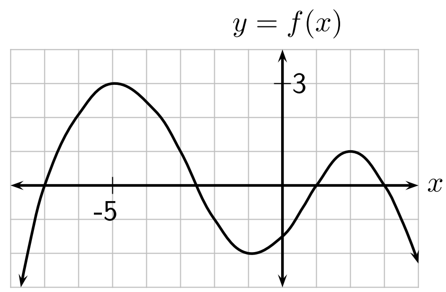
9. Consider the function $g(x) = x^2 + 5x$.

   (a) Find and simplify $g(a-3)$ by substituting $a-3$ for x and simplifying. *From this point on you will always be expected to simplify, even if not specifically asked to do so.*

   (b) Find and simplify $g(x+h)$.

   (c) Use your result from (b) to find and simplify $\dfrac{g(x+h) - g(x)}{(x+h) - x}$. *This sort of expression is very important in mathematics and its applications!* We'll see what it is about later.

10. Use the function shown to the right to answer the following questions.



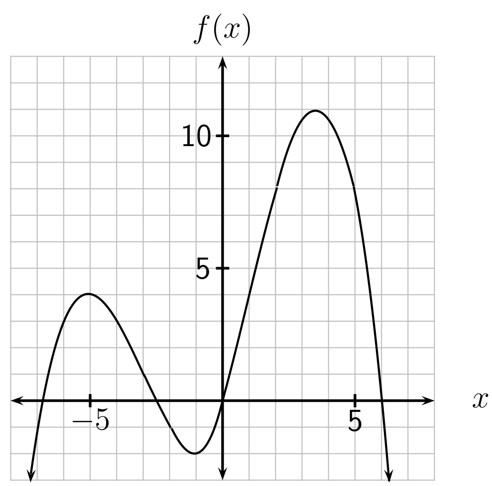
   (a) Give all of the x- and y-intercepts for the function f. Distinguish between the two!

   (b) Estimate the values of $f(-3)$, $f(-2)$ and $f(3)$.

   (c) Estimate *all* values of x for which $f(x) = 2$.

   (d) Estimate *all* values of x for which $f(x) = 3$.

11. Consider again the graph of the function f from Example 2.1(e), shown below and to the right. Use the graph to do each of the following:



   (a) Find $f(-3)$, $f(-1)$, $f(1)$.

   (b) Find x such that $f(x) = 1$. (There are four values of x for which this happens - give them all.)

   (c) Find $f(0)$.

   (d) Find all x such that $f(x) = 4$.

   (e) Find all x such that $f(x) = 10$.

12. We sometimes also want to know what input values can cause two different functions to have the same output values. Consider again $f(x) = 3x - 8$, along with $g(x) = x^2 - 3x$.

    (a) Find $f(4)$ and $g(4)$. They should be equal.

    (b) The question here is "How do we know that $x = 4$ will make the two functions equal?" (That is, make them have the same output.) Well, we want a value of x that makes $f(x) = g(x)$, so we set $3x - 8 = x^2 - 3x$ and solve. Do this.

    (c) You should have obtained $x = 4$ and $x =$ _____ in part (b). Substitute the other value that you found into both functions and verify that it does give the same output for both.

13. Find all values of t for which $r(t) = \sqrt{2t+5}$ and $s(t) = t + 1$ are equal. *Be sure to check your answer(s) in the original functions.*

14. Find all values of y for which $f(y) = \dfrac{y+3}{y^2 - y}$ and $g(y) = \dfrac{8}{y^2 - 1}$ are equal.

15. Consider the function $f(x) = 3x^4 - 5x^2$.

    (a) Find $f(-2)$ and $f(2)$, then find $f(-5)$ and $f(5)$.

    (b) Find *and simplify* $f(-x)$. What do you notice, in light of your answers to (a)? A function that behaves in this way is called an **even function**.

16. Now let $g(x) = x^3 + 5x$.

    (a) The function could be written as $g(x) = x^3 + 5x^r$ for what value of r? What kind of function do you suppose that this one is?

    (b) Find $f(-1)$ and $f(1)$, then $f(-4)$ and $f(4)$.

    (c) What can we say in general about $f(-x)$ for this function?

17. In light of the previous two exercises, what can you say about each of the functions

    (a) $h(x) = 12$ (See Exercise 6.) (b) $f(x) = x^2 - 5x + 2$

18. Find and simplify $f(x+h)$ for $f(x) = x^2 - 5x + 2$.

19. Find and simplify $g(x+h)$ for $g(x) = 3x - x^2$.

2.2 Domain and Range of a Function

Performance Criteria:

2. (c) Describe a set using set builder notation or interval notation.
   (d) Give the domain and range of a function from the equation of the function.
   (e) Give the domain and range of a function from its graph.

Consider the function $f(x) = \sqrt{25 - x^2}$. We can see that

$$f(0) = \sqrt{25 - 0^2} = 5, \qquad f(-5) = \sqrt{25 - (-5)^2} = 0, \qquad f(4) = \sqrt{25 - 4^2} = 3.$$

We can also see that $f(6) = \sqrt{25 - 6^2} = \sqrt{-11}$ is not possible to find unless we allow complex numbers, which we are not doing now. Because the value of x is squared in this function, $f(-6)$ results in the same situation, and is also not possible. In fact, we can only find $f(x)$ for values of x between -5 and 5, including both of those numbers. We call the collection of those numbers for which $f(x)$ can be found the **domain** of f. In this section we first look at how to describe such sets concisely, and then we look at the domains of some other functions.

Sets of Real Numbers

A **set** is a collection of objects, called **elements**; in our case the elements will always be numbers. Sets will be important at times in our study of functions, and it is helpful to have clear, concise ways of describing sets. When we have a set of just a few numbers, we can describe the set by listing its elements (generally listed from smallest to largest) and enclosing them with the symbols $\{$ and $\}$, used to indicate a set. For example, the set consisting of the numbers -1, 3 and 10 would be denoted by $\{-1, 3, 10\}$.

It is occasionally useful to consider sets with just a few elements, and even sets with just one element, like $\{5\}$. More often, however, we will consider sets that cannot be listed, like the numbers between -5 and 5, including both. When we say "all the numbers" we mean all the **real numbers**, which means every number that is not a complex number. The real numbers consist of all whole numbers and fractions, as well as more unusual numbers like $\sqrt{2}$ and π (and negatives of all these things as well). All the real numbers taken together is a set, which we denote with the symbol $\mathbb{R}$.

To describe a set like the set of all real numbers between -8 and 3, for example, we will usually use one of two methods. The first we'll describe is called **set builder notation**. The set of all real numbers between -8 and 3, including -8 and not including 3 is described in set builder notation as

$$\{x \in \mathbb{R} \mid -8 \leq x < 3\} \qquad \text{or} \qquad \{x \mid -8 \leq x < 3\}$$

The symbol $\in$ means "in the set" and the vertical bar represents the words "such that." The first statement above is then read as "the set of real numbers x such that x is greater than or equal to -8 and less than 3." Since we assume we are considering only real numbers unless told otherwise, we can eliminate the $\in \mathbb{R}$, which means "x is an element of the set of real numbers." For a set like all numbers greater than or equal to -2 the set builder notation would be $\{x \mid x \geq -2\}$

⋄ **Example 2.2(a):** Give the set builder notation for each of the following sets:

(a) The set of all numbers between -5 and 3, not including -5 but including 3.

(b) The set of all numbers between 1 and 3, including both.

(c) The set of all numbers less than -2.

Solution:

(a) $\{x \mid -5 < x \leq 3\}$ (b) $\{x \mid 1 \leq x \leq 3\}$ (c) $\{x \mid x < -2\}$

Interval Notation for Sets of Real Numbers

Let's go back to the set of all real numbers between -8 and 3. Another method for describing sets like those that we have been looking at is called **interval notation**. This consists of writing the smaller of the two numbers bounding our set, followed by the larger: $-8, 3$. We then enclose these numbers with square brackets [ ] or parentheses ( ), depending on whether the numbers themselves are actually included in the set or not, respectively. For example, if -8 was to be included but 3 was not, we would write $[-8, 3)$. If we meant only the numbers between -8 and 3, but not including either, we would write $(-8, 3)$. If both -8 and 3 were to be included we'd write $[-8, 3]$.

For a set like all real numbers greater than -2, we need to have the concept of infinity, symbolized as ∞. And along with infinity comes negative infinity! Think of them this way: Neither of them is a number, so they cannot be included in sets. Infinity should be thought of as a "place" that is bigger than all real numbers, and negative infinity is a place that is smaller than all real numbers. Negative infinity is symbolized as $-\infty$. We use the concepts of positive and negative infinity like this: The set of all real numbers greater than or equal to -2 is written as $[-2, \infty)$. The set of all real numbers less than -2 is written as $(-\infty, -2)$. *Note that neither infinity or negative infinity can ever be included in a set, since neither is a number. Thus they are always enclosed by parentheses.*

⋄ **Example 2.2(b):** Give the interval notation for each of the following sets:

(a) The set of all numbers between -5 and 3, not including -5 but including 3.

(b) The set of all numbers between 1 and 3, including both.

(c) The set of all numbers less than or equal to 4.

Solution:

(a) $(-5, 3]$ (b) $[1, 3]$ (c) $(-\infty, 4]$

Describing Other Sets of Numbers

Suppose that instead of all the numbers between -100 and 100 we needed to consider all the numbers less than or equal to -100 along with all those greater than or equal to 100. In other words, we want the set consisting of $(-\infty,-100]$ and $[100,\infty)$, combined. We call such a combination the **union** of the two sets, symbolized by $(-\infty,-100]\cup[100,\infty)$. Taken as one set, this is the set of all numbers that are less than or equal to -100 *OR* greater than or equal to 100. The same set can be described with set builder notation as $\{x \mid x \leq -100 \text{ or } x \geq 100\}$. Note that we don't use the word *and* because there are no numbers that are less than -100 and at the same time greater than 100. Another way of using set builder notation to describe this set would be $\{x \mid x \leq -100\}\cup\{x \mid x \geq 100\}$

◇ **Example 2.2(c):** Describe the set of all numbers less than -2 or greater than 2 using (a) set builder notation, and (b) interval notation.

Solution: The set is described with set builder notation by either of $\{x \mid x < -2 \text{ or } x > 2\}$ or $\{x \mid x < -2\}\cup\{x \mid x > 2\}$, and with interval notation by $(-\infty,-2)\cup(2,\infty)$.

Suppose that we are asked to describe the set of all real numbers not equal to seven. Although it is possible to use interval notation to describe this set, it is actually much easier to do using set builder notation: $\{x \mid x \neq 7\}$. If we wanted to describe the set of all real numbers not equal to 7 or 2, we would write $\{x \mid x \neq 2 \text{ and } x \neq 7\}$. Notice that we must use the word *and* in order to be logically correct here. Often we write instead $\{x \mid x \neq 2,\ 7\}$, which means the same thing.

Domains of Functions

Recall the function $f(x)=\sqrt{25-x^2}$ that we began this section with. We can now use set builder notation or interval notation to say that its domain is $\{x \mid -5 \leq x \leq 5\}$ or $[-5,5]$. To tell that this set is the domain of the function f we use the notation Dom$(f)=[-5,5]$, spoken as "the domain of f is the set ..." Consider the two functions $g(x)=x^2+5x$ and $h(x)=\dfrac{3}{x-7}$. It should be clear to you that we can compute $g(x)$ for *any* choice of x, so Dom$(g)=\mathbb{R}$. We run into trouble when we try to compute $h(7)$. We *can* compute $h(x)$ for any other value of x, however, so Dom$(h)=\{x \mid x \neq 7\}$.

For the time being, there are only two things that can cause a problem with computing the value of a function:

- Having zero in the denominator of a fraction.
- Having a negative value under a square root (or any other *even* root).

◇ **Example 2.2(d):** Give the domain of the function $g(x)=\sqrt{x-3}$ using both set builder notation and interval notation.

Solution: To avoid having a negative under the square root, we must be sure that x is at least 3. Thus

$$\mathrm{Dom}(g)=\{x \mid x \geq 3\} \qquad \text{or} \qquad \mathrm{Dom}(g)=[3,\infty)$$

◇ **Example 2.2(e):** Give the domain of the function $f(x) = \dfrac{x-1}{x^2+x-20}$ using set builder notation or interval notation, whichever is more appropriate.

Solution: The denominator of f factors to give us $f(x) = \dfrac{x-1}{(x+5)(x-4)}$, so x cannot be either -5 or 4. Therefore $\text{Dom}(f) = \{x \mid x \neq -5,\ 4\}$. Note that there is no problem with x having the value 1, since that causes a zero in the numerator, which is not a problem. (It does mean that $f(1) = 0$.)

◇ **Example 2.2(f):** Give the domain of the function $h(x) = \dfrac{1}{\sqrt{x^2-16}}$ using interval notation.

Solution: To avoid having a negative under the square root, we must be sure that x is at least 4, but we also need to make sure we don't get a zero in the denominator, so x cannot be 4. We can see that x can also be less than -4 because then x^2 is greater than 16 and the square root will be defined. Thus $\text{Dom}(h) = (-\infty,-4)\cup(4,\infty)$.

Ranges of Functions

There is another concept that goes along with the domain of a function. Consider the function $f(x) = x^2+3$, whose domain is $\mathbb{R}$. Suppose that we were to find $f(x)$ for every real number - what values would we get? Well, the possible values of x^2 are all the positive real numbers and zero, or the set $[0,\infty)$. When we add three to all of those values, we will get every number greater than or equal to 3, or the set $[3,\infty)$. The set of all possible outputs of a function (when it is evaluated for every number in its domain) is called the **range** of the function, abbreviated Ran. So, for our example, Ran$(f) = [3,\infty)$. Ranges are generally more difficult to determine than domains.

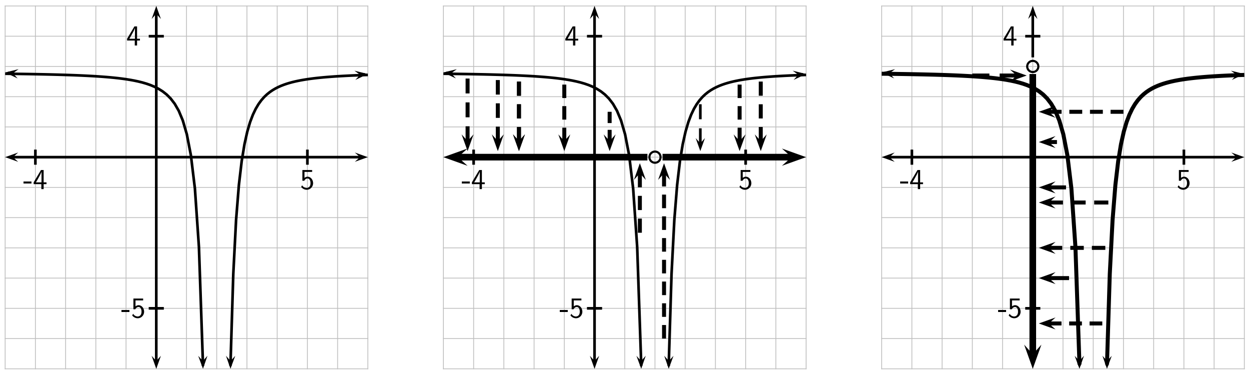
◇ **Example 2.2(g):** Find the range of $g(x) = \sqrt{x+5}-2$.

Solution: We first note that $\text{Dom}(g) = [-5,\infty)$. When $x=-5$, $\sqrt{x+5} = 0$ and, as x gets larger, $\sqrt{x+5}$ gets larger as well, getting as large as we want. Thus the range of $\sqrt{x+5}$ is $[0,\infty)$. When we then subtract 2 from all those values we get $\text{Ran}(g) = [-2,\infty)$.

Domains and Ranges, Graphically

It is relatively simple to determine the domain and range of a function if we have its graph. As an example, let's consider the function $h(x) = \dfrac{3x^2-12x+11}{(x-2)^2}$, whose graph is shown below and to the left. To find the domain of the function we simply need to find all the points on the x-axis that correspond to a point on the curve. In the middle picture below I have tried to indicate how we "drop" or "raise" every point onto the x-axis. I have darkened the part of the

x-axis that is hit by points coming from the curve like this; it is the entire axis except the point $(2, 0)$. (It should be clear from the equation why this is!) Thus the domain is $\{x \mid x \neq 2\}$.

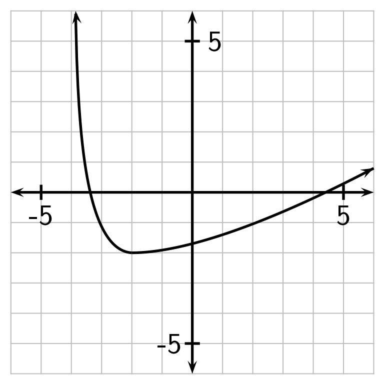


To determine the range we go horizontally to the y-axis from every point on the curve, as shown in the third picture above. The points "hit" are the ones on the y-axis that are shaded in that picture, consisting of the set $(-\infty, 3)$, so that is the range of the function.

◇ **Example 2.2(h):** Give the domain and range of the function whose graph is shown below and to the right. The left edge of the graph gets closer and closer to the vertical line $x = -4$ without ever touching it.

Solution: We can see that if we were to thicken the x-axis at all points above or below points on the curve, we would thicken the portion to the right of $x = -4$, but not -4 itself. the domain of the function is then $(-4, \infty)$.

If we instead thicken the y-axis at all points that are horizontal from some point on the curve, we will thicken the part from $y = -1$ up, including -1. Therefore the range of the function is $[-1, \infty)$.



Of course the set builder notation for the domain would be $\{x \mid x > -4\}$ and for the range it would be $\{y \mid y \geq -1\}$. (Note the use of the correct variable for each.)

Section 2.2 Exercises To Solutions

1. Give the set builder notation for each of the following sets.

   (a) All real numbers between $2\frac{1}{2}$ and 13, including both.

   (b) All real numbers between -12 and -2, including -2 but not -12.

   (c) All real numbers between -100 and 100, including neither.

(d) All real numbers greater than 3.

(e) All real numbers less than or equal to -5.

2. Give the interval notation for each of the sets described in Exercise 1.

3. Use interval notation and union to write each of the following sets.

   (a) The real numbers less than 3 or greater than or equal to 7.

   (b) The real numbers less than -4 or greater than 4.

   (c) The real numbers not equal to 7. (Note that this is the set of real numbers less than 7 or greater than 7.)

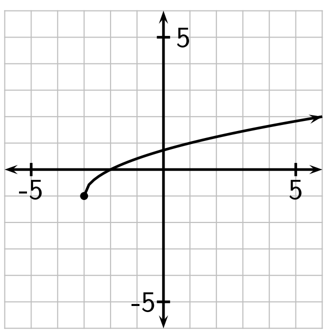
4. Use set builder notation to write each of the sets in the previous exercise.

5. Give the domain of each of the following functions using either interval notation or set builder notation.

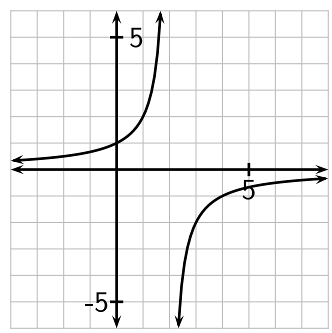
   (a) $f(x) = \sqrt{x+2}$ (b) $g(x) = \dfrac{4}{x-1}$ (c) $h(x) = \dfrac{x-2}{x^2+3x-10}$

6. For each graph, give the domain and range of the function whose graph is shown.

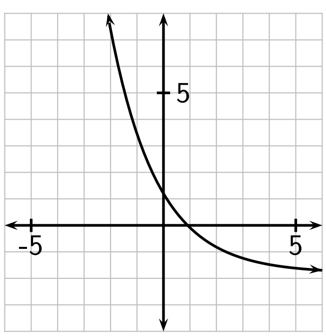
(a)



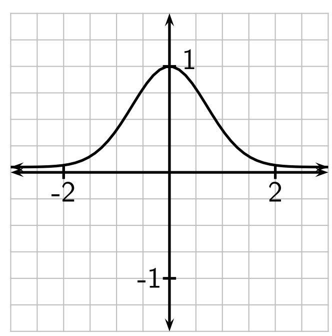
(b)



(c)



(d)



7. For each of the following,

   - attempt to determine the domain of the function algebraically
   - give the domain using either interval notation or set builder notation
   - Graph the function using technology, and ee if the graph seems to support your answer

- check your answer in the back of the book

(a) $y = \sqrt{x-4}$ (b) $f(x) = \sqrt{4-x}$ (c) $g(x) = \dfrac{5}{x+3}$

(d) $y = \sqrt{25-x^2}$ (e) $g(x) = \dfrac{x-1}{x+5}$ (f) $f(x) = \dfrac{7}{3x-2}$

(g) $y = \dfrac{5}{x^2-5x+4}$ (h) $f(x) = \dfrac{\sqrt{x-1}}{x-5}$ (i) $g(x) = x^2 - 5x + 4$

8. (a) Give the domain of $f(x) = \sqrt{x-3}$ using both set builder notation and interval notation.
   (b) Give the domain of $g(x) = \sqrt{3-x}$, noting how your answer compares to that of (a).
   (c) The domain of $h(x) = \dfrac{1}{\sqrt{3-x}}$ is slightly different than the domain of $g(x) = \sqrt{3-x}$. How is it different, and why? Give it.

9. (a) What is the domain of any function of the form $f(x) = ax^2 + bx + c$?
   (b) Suppose that the graph of a function is a parabola that has vertex $(-2, 5)$ and opens downward. Give the range of the function.
   (c) Determine the range of $y = x^2 + 4x - 5$ without graphing the function.

10. Use the following questions to help you determine the range of the function $y = \dfrac{x^2}{x^2+3}$.

    - Can the function ever be negative?
    - Can the function ever be positive?
    - Can the function ever be zero?
    - How does the size of the numerator compare with the size of the denominator? Can they ever be equal?
    - Write down what you think the range of the function is. If the above questions are not enough for you to determine the range, substitute some values for x and find the corresponding values of y.
    - Graph the function using technology and see if the graph supports your answer.

11. For each of the following given sets, determine a function whose domain is the set.

(a) $[-2, \infty)$ (b) $\{x \mid x \neq 5\}$ (c) $(-\infty, 5]$

(d) $\{x \mid x \neq -5, 1\}$ (e) $(-\infty, -3)$

2.3 Behaviors of Functions

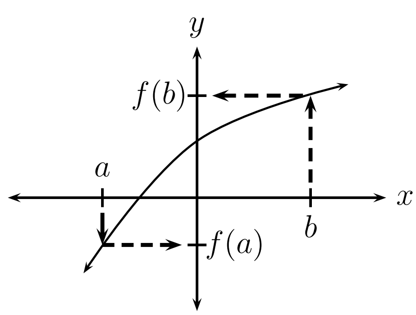
Performance Criteria:

2. (f) Given the graph of a function,
   - give the intervals on which the function is increasing (decreasing)
   - give the intervals on which the function is positive (negative)
   - give the maxima and minima of the function and their locations, and determine whether each is relative or absolute
   - identify the function as even, odd, or neither

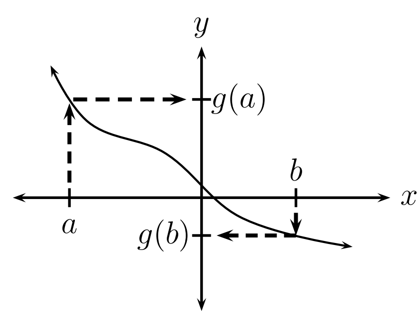
   (g) Given the definitions of even and odd functions, determine whether a function is even, odd or neither from its equation.

Increasing and Decreasing Functions

The graphs of two functions f and g are shown below. Note the following difference between the two functions: When we "input" two values a and b into f, if $a < b$ then $f(a) < f(b)$. With g, on the other hand, if $a < b$, then $g(a) > g(b)$. Put more simply, as x gets larger $f(x)$ gets larger, and $g(x)$ gets smaller as x gets larger. Graphically, as we move to the right on the graph of f we move up, and as we move to the right on the graph of g we move down.



$f(x)$

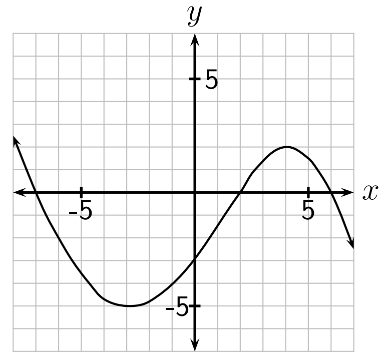


$g(x)$

We call f an **increasing function** and g a **decreasing function**. To summarize:

- A function is **increasing** if increasing the input causes an increase in the output.
- A function is **decreasing** if increasing the input causes a *decrease* in the output.

Many functions that we will deal with are neither increasing or decreasing. Instead, they are increasing for some values of x, and decreasing for others. For example, the function h shown to the right at the top of the next page is increasing for values of x between -3 and 4. Using interval notation, we say the function is increasing on the interval $(-3, 4)$ or on the interval $[-3, 4]$. It doesn't really matter whether we include the endpoints of the interval or not, for technical reasons that I don't wish to go into here. Similarly, h is decreasing on the intervals $(-\infty, -3)$ and $(4, \infty)$ or $(-\infty, -3]$ and $[4, \infty)$. (We assume that the behavior we can't see, beyond the edges of the grid, continues in the manner indicated by the arrows on the ends of the graph.)

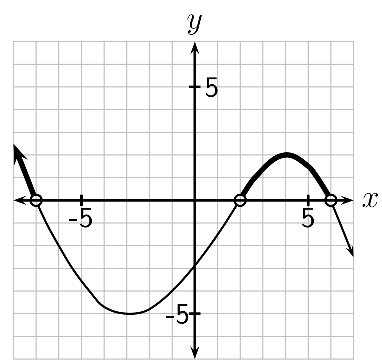


The function $h(x)$

Notice that when we talk about where a function is increasing or decreasing, we mean the x values, NOT the y values. For the idea of increasing and decreasing, and the others we are about to see, you might wish to think this way: The graph of the function is like a landscape, with hills and valleys. The x-axis is like a map, and every point on the landscape has a map location given by its x-coordinate. The y-coordinate of a point is its elevation. When we ask where a function is increasing, *we mean the "map locations,"* or the x-values.

Positivity and Negativity of a Function

We will often be interested in where a function is positive or negative. The function is the y values, so we want to know where y is positive on the graph of the function. The "where" part is given by x values, in the same way as done for increasing and decreasing. The graph of the function h is shown again, to the right, but this time the portions of the curve where y is positive are thickened. The"empty holes" where the function crosses the x-axis are to indicate that the y values are not positive at those points. To tell where the function is positive we give the x values corresponding to all the points on those thickened parts of the curve. Thus we say that the function is positive on the intervals $(-\infty, -7)$ and $(2, 6)$. Obviously the function is then negative on the intervals $(-7, 2)$ and $(6, \infty)$.



The function $h(x)$

Minima and Maxima of a Function

Looking at the graph of h again, we can see that the curve has a high point at $(4, 2)$. We call the value 2 a **maximum** of the function, and when talking about a maximum we want to identify two things:

i) *What the maximum value is*; this means the output, or y, value.

ii) *Where the maximum value occurs*; this means the x value for the point with the maximum y value.

In this case we would say that h has a maximum of 2 at $x = 4$. Note that the plural of maximum is *maxima*, and the plural of minimum is *minima*.

One can see that the point at $(4, 2)$ is not actually the highest point on the graph, since every point on the part of the graph to the left of about -7.8 has a y value greater than two. However, the point at $(4, 2)$ is higher than all the points on the curve that are near that point, so we say that the function has a **relative maximum** there. A relative maximum of a function occurs at a value of $x = a$ if $f(a) > f(x)$ for all values of x near a. (Take a minute and try to digest what this says!) On a graph, relative maxima occur at the tops of "hills" on the graph. Similarly, a relative minimum of a function occurs at a value of $x = b$ if $f(b) < f(x)$ for all values of x near b. The function h has a relative minimum of -5 at $x = -3$.

Any value $x = a$ such that $f(a) > f(x)$ for *ALL* other values of x is the location of what we call an **absolute maximum**. On a graph, this shows up as a "hill" that is higher than any other point on the graph, including points not actually shown on the picture. The function h has no absolute maximum (or absolute minimum). *Note that any absolute maximum is "automatically" a relative maximum as well.*

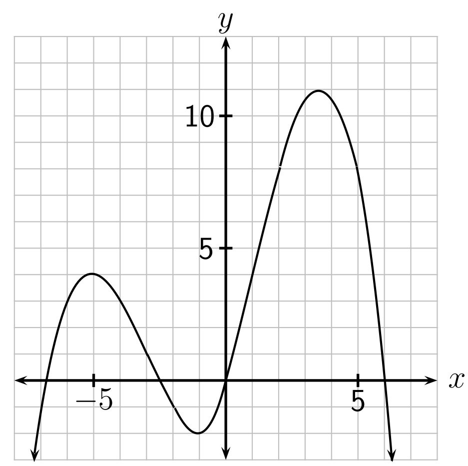
◇ **Example 2.3(a):** For the graph of the function f shown to the right, give

(a) the intervals on which the function is decreasing,

(b) the intervals on which the function is positive,

(c) all maxima and minima, including their locations and whether they are relative or absolute.

It will be necessary to approximate some values of x and y.



Solution: The function is decreasing on the intervals $(-5, -1)$ and $(3.5, \infty)$. It would also be correct to say instead that f is decreasing on $[-5, -1]$ and $[3.5, \infty)$. The function is positive on the intervals $(-6.8, -2.5)$ and $(0, 6)$, with the first interval being approximate. Note that $x = 0$ MUST be excluded because $f(0)$ is not positive.

The function has the following maxima and minima:

– an absolute maximum of 11 at about $x = 3.5$

– a relative minimum of -2 at $x = -1$

– a relative maximum of 4 at $x = -5$

The function has no absolute minimum.

Even and Odd Functions

As you progress through these notes you will study various specific types of functions, like linear and quadratic functions, polynomial functions, rational functions, and exponential and logarithmic functions. Most of you will go on to take a course in trigonometry, in which you will study trigonometric functions. Overlapping all those kinds of functions, with the exception of exponential and logarithmic functions are two kinds of functions called even and odd functions. What I mean by "overlapping" is that linear, quadratic, polynomial, rational and trigonometric functions can also be even functions, odd functions, or they can be neither even nor odd.

Let's begin with the definitions of even and odd functions:

Even and Odd Functions

- $f(x)$ is said to be an **even function** if $f(-x) = f(x)$ for all x in the domain of f.
- $f(x)$ is said to be an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f.

What is this saying? A function is even if putting in a value or its opposite (negative) both result in the same value. The classic example is $f(x) = x^2$; if we put in, say, 5 or -5 we get 25 out either way, and the same goes for all such pairs of values. More formally, to test whether a function is even, we evaluate it for $-x$ and see if the result is $f(x)$

◇ **Example 2.3(b):** Determine whether the function $g(x) = 3x^4 - x^2 + 6$ is even.

Another Example

Solution: Because $g(-x) = 3(-x)^4 - (-x)^2 + 6 = 3x^4 - x^2 + 6 = g(x)$, the function is even.

A function consisting whole number powers of x, each multiplied by a constant and all added or subtracted, is called a **polynomial function**; we will study such functions in more detail later. The function g is a polynomial function, and note that it can be written as $g(x) = 3x^4 - x^2 + 6x^0$, since $x^0 = 1$. Zero is considered an even number; note that all of the exponents in g are even, and g is an even function. As you might then guess, any polynomial function in which all of the exponents are odd, like $h(x) = x^3 - 7x$, is an odd function. This can be seen by the computation

$$h(-x) = (-x)^3 - 7(-x) = -x^3 + 7x = -(x^3 - 7x) = -h(x).$$

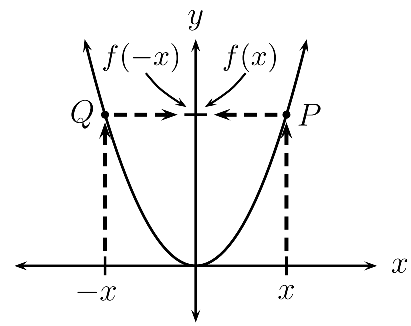
Not all functions are even or odd - in fact, "most" functions are neither even nor odd. An example of a function that is neither even or odd would be $f(x) = x^2 - 5x + 2$. Note that

$$f(-x) = (-x)^2 - 5(-x) + 2 = x^2 + 5x + 2 \neq f(x) \quad \text{and} \quad -f(x) = -x^2 + 5x - 2 \neq f(-x)$$

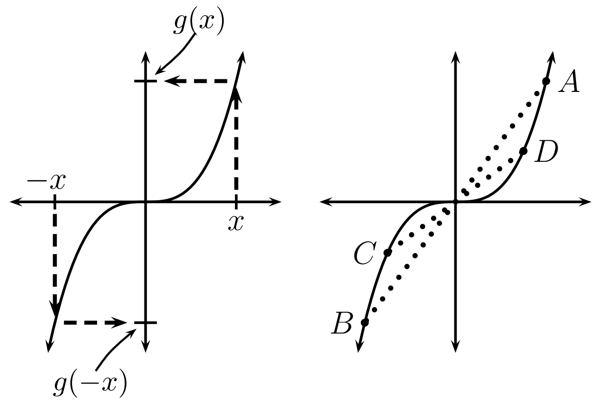
Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, f is neither even nor odd.

Graphs of Even and Odd Functions

Graphs of even and odd functions demonstrate special kinds of "balance" that we call **symmetry**. The graph of an even function is *symmetric with respect to the y axis*, which means that for every point on the graph there is a point on the opposite side of the y-axis and the same distance away. We illustrate this to the right using the function $f(x) = x^2$, an even function. Note that inputs of x and $-x$ both give us the same output; the points P and Q are symmetric with respect to the y-axis.



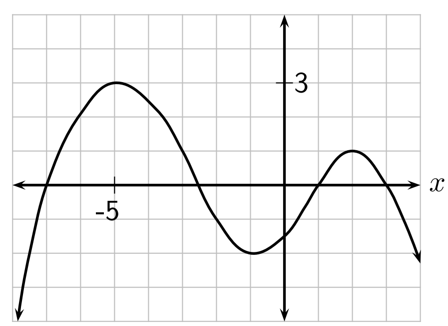
The two graphs to the right are for the function $g(x) = x^3$, an odd function. The first graph illustrates the definition of an odd function: If we input the opposite values x and $-x$ (seen as opposites on the x-axis), the corresponding outputs $g(x)$ and $g(-x)$ can be seen to be opposites on the y-axis. This is stated mathematically by $g(-x) = -g(x)$. The second graph illustrates the kind of symmetry that odd functions have, *symmetry with respect to the origin*. That is, for each point on the curve there is a corresponding point on the opposite side of the origin on a line through the origin, and the same distance from the origin. In the picture shown, the point A corresponds to B, and C corresponds to D.



Section 2.3 Exercises

To Solutions

1. For the function f, whose graph is shown below and to the right, answer the following questions. Both "tails" of the graph continue to spread outward and downward.



   (a) Give all of the x-intercepts and y-intercepts for the function f. Distinguish between the two!

   (b) Give the domain of the function.

   (c) Give the range of the function.

   (d) Give the intervals on which f is increasing.

   (e) Give the intervals on which f is negative.

   (f) Give all relative maxima and minima in the manner used in Example 2.3(a).

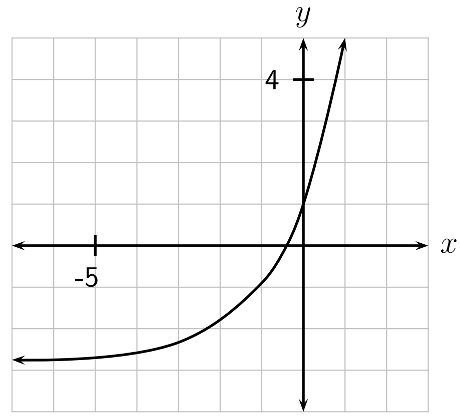
   (g) Does f have an absolute maximum? If so, what is it, and at what x value (or values) does it occur?

   (h) Does f have an absolute minimum? If so, what is it and at what x value (or values) does it occur?

(i) Estimate the value of $f(-4)$.

(j) Estimate all values of x where $f(x) = -1$.

2. For the function g, whose graph is shown below and to the right, answer the following questions. The right tail continues upward and rightward forever. The left tail continues leftward with y values getting closer and closer to, but never reaching, -3.



(a) Give the domain of the function.

(b) Give the range of the function.

(c) Estimate the values of $g(-2)$ and $g(-3)$.

(d) Estimate *all* values of x for which $g(x) = -1$.

(e) Estimate *all* values of x for which $g(x) = 5$.

(f) Give the intervals on which g is increasing.

(g) At what x values does the function g have relative maxima or minima? Answer with appropriate statements.

(h) Give the intervals on which the function is positive.

3. Essential to determining from the definition whether a function is even, odd or neither are the facts that

$$(-x)^n = x^n \text{ if } n \text{ is even, and } (-x)^n = -x^n \text{ if } n \text{ is odd.}$$

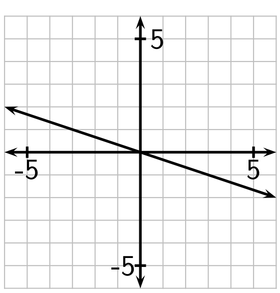
For each of the following functions, calculate $f(-x)$ by substituting $-x$ for x and using the above two facts. If the result is not $f(x)$ itself (indicating an even function), see if you can factor out a negative to get $-f(x)$, indicating an odd function. Conclude by telling whether the function is even, odd or neither.

(a) $f(x) = 3x + 1$ (b) $f(x) = x^2 - 5$ (c) $f(x) = 5x^3 - 7x + 1$

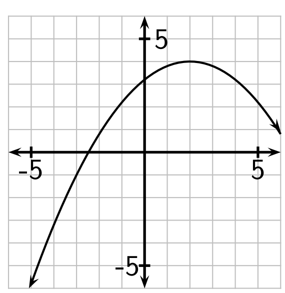
(d) $f(x) = 10$ (e) $f(x) = x^5 + 4x$

4. Each graph below is that of a function. For each, determine whether the function is odd, even, or neither.

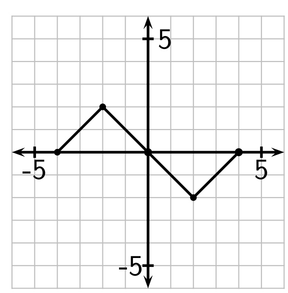
(a)



(b)



(c)



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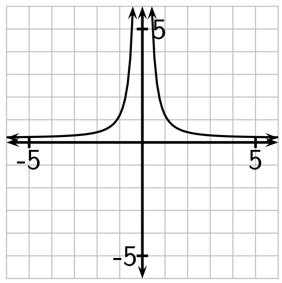

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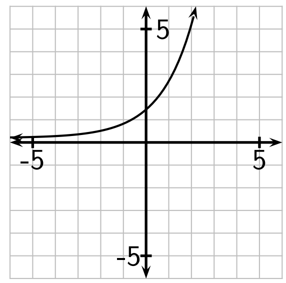

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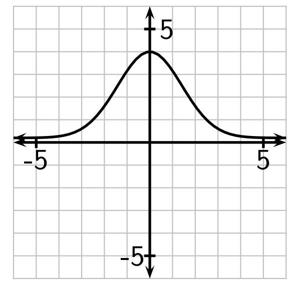


5. Discuss the existence of relative and absolute maxima and/or minima for quadratic functions.

6. Consider the function $g(x) = -x^2 + 2x + 3$. Do all of the following without actually graphing the function. (You may wish to think about the appearance of the grph, though.)

   (a) What sort of minima or maxima do we know the function has to have?

   (b) Give the value of any maximum or minimum, where it occurs and what kind of maximum or minimum it is.

   (c) Where is the function decreasing?

   (d) Where is the function positive?

   (e) What is the domain of the function?

   (f) What is the range of the function?

   (g) Use technology to graph the function, and use the graph to check your answers to (a)-(d).

7. Suppose that the function h is odd.

   (a) If $h(3) = -7$, what is $h(-3)$? (b) If $h(5) = 12$, then $h($ _____ $) =$ _____

8. (a) Suppose that $f(4) = 7$ and $f(-4) = 7$ for some function f. Knowing only that, can we say for sure whether or not f is even? Explain.

   (b) For another function g we know that $g(-2) = 5$ and $g(2) = -7$. Can we say for sure whether or not g is odd? Explain.

9. Most of you will take trigonometry, where you will work with two functions called the **sine** and **cosine** functions. When we write $y = \sin x$, the letters sin are like the letter f for a function $f(x)$. So sine is like the box shown on the first page of Section 2.1 illustrating a function as a "machine." With the sine function we can find out what the machine does to a number by simply computing the sine of the number using the sin button on our calculators. The same holds for the cosine function.

   (a) Use your calculator to find $\sin x$ for several positive and negative pairs of the same number. Does the sine function appear to be even, odd, or neither?

(b) Use technology to graph $y = \sin x$. Does the graph support your answer to (a)?

(c) Repeat part (a) for the cosine function.

(d) Use technology to graph $y = \cos x$. Does the graph support your answer to (c)?

10. Sketch the graph of a function that is

- increasing on the intervals $(-\infty, -1)$ and $(2, 3)$ and decreasing everywhere else
- positive on $(-2, 1)$, zero at $x = -2$ and $x = 1$, and negative everywhere else.

11. Suppose that all of the conditions listed in Exercise 10 hold except that the function is also zero at $x = 3$. What sorts of minima and maxima (relative? absolute?) does the function have, and where do they occur? (One location is exact, the other two can only be determined to be in certain intervals - give the intervals in those cases.

12. Use your calculator to graph the function $f(x) = x^4 - 2x^3 - 5x^2 + 6x$. Then write a short paragraph about the function, addressing all of the concepts that have been discussed in this section.

2.4 Mathematical Models: Linear Functions

Outcome/Criteria:

2. (h) Use a linear model to solve problems; create a linear model for a given situation.
   (i) Interpret the slope and intercept of a linear model.

Using a Linear Model to Solve Problems

An insurance company collects data on amounts of damage (in dollars) sustained by houses that have caught on fire in a small rural community. Based on their data they determine that the expected amount D of fire damage (in dollars) is related to the distance d (in miles) of the house from the fire station. (Note here the importance of distinguishing between upper case variables and lower case variables!) The equation that seems to model the situation well is

$$D = 28000 + 9000d$$

This tells us that the damage D is a function of the distance d of the house from the fire station. Given a value for either of these variables, we can find the value of the other.

◇ **Example 2.4(a):** Determine the expected amount of damage from a house fire that is 3.2 miles from the fire station, 4.2 miles from the fire station and 5.2 miles from the fire station.

Solution: For convenience, let's rewrite the equation using function notation, and in the slope-intercept form: $D(d) = 9000d + 28000$. Using this we have

$$D(3.2) = 9000(3.2) + 28000 = 56800, \qquad D(4.2) = 65800, \qquad D(5.2) = 74800$$

The damages for distances of 3.2, 4.2 and 5.2 miles from the fire station are $\$56,800$, $\$65,800$ and $\$74,800$.

Note that in the above example, for each additional mile away from the fire station, the amount of damage increased by $9000.

◇ **Example 2.4(b):** If a house fire caused $47,000 damage, how far would you expect that the fire might have been from the fire station? *Round to the nearest tenth of a mile.*

Solution: Here we are given a value for D and asked to find a value of d. We do this by substituting the given value of D into the equation and solving for d:

$$\begin{aligned} 47000 &= 9000d + 28000 \\ 19000 &= 9000d \\ 2.1 &= d \end{aligned}$$

We would expect the house that caught fire to be about 2.1 miles from the fire station.

◇ **Example 2.4(c):** How much damage might you expect if your house was right next door to the fire station?

Solution: A house that is right next door to the fire station is essentially a distance of zero miles away. We would then expect the damage to be

$$D(0) = 9000(0) + 28000 = 28000.$$

The damage to the house would be $\$28,000$.

Interpreting the Slope and Intercept of a Linear Model

There are a few important things we want to glean from the above examples.

- When we are given a mathematical relationship between two variables, if we know one we can find the other.
- Recall that slope is rise over run. In this case rise is damage, measured in units of dollars, and the run is distance, measured in miles. Therefore the slope is measured in $\frac{\text{dollars}}{\text{miles}}$, or dollars per mile. The slope of 9000 dollars per mile tells us that for each additional mile farther from the fire station that a house fire is, the amount of damage is expected to increase by $9000.
- The amount of damage expected for a house fire that is essentially right at the station is $28,000, which is the D-intercept for the equation.

In general we have the following.

> **Interpreting Slopes and Intercepts of Lines**
>
> When an "output" variable depends linearly on another "input" variable,
>
> - the slope has units of the output variable units over the input variable units, and it represents the amount of increase (or decrease, if it is negative) in the output variable for each one unit increase in the input variable,
> - the output variable intercept ("y"-intercept) is the value of the output variable when the value of the input variable is zero, and its units are the units of the output variable. *The intercept is not always meaningful.*

The first of these two points illustrates what was pointed out after Example 2.4(a). As the distance increased by one mile from 3.2 miles to 4.2 miles the damage increased by $65800 - 56800 = 9000$ dollars, and when the distance increased again by a mile from 4.2 miles to 5.2 miles the damage again increased by $\$9000$.

When dealing with functions we often call the "input" variable (which is *ALWAYS* graphed on on the horizontal axis) the **independent variable**, and the "output" variable (which is always graphed on the vertical axis) is the **dependent variable**. Using this language we can reword the items in the previous box as follows.

Slope and Intercept in Applications

For a linear model $y = mx + b$,

- the slope m tells us the amount of increase in the dependent variable for every one unit increase in the independent variable
- the vertical axis intercept tells us the value of the dependent variable when the independent variable is zero

Section 2.4 Exercises

To Solutions

1. The amount of fuel consumed by a car is a function of how fast the car is traveling, amongst other things. For a particular model of car this relationship can be modeled mathematically by the equation $M = 18.3 - 0.02s$, where M is the mileage (in miles per gallon) and s is the speed (in miles per hour). *This equation is only valid when the car is traveling on the "open road" (no stopping and starting) and at speeds between 30 mph and 100 mph.*

   (a) Find the mileage at a speed of 45 miles per hour and at 50 miles per hour. Did the mileage increase, or did it decrease, as the speed was increased from 45 mph to 50 mph? By how much?

   (b) Repeat (a) for 90 and 95 mph.

   (c) Give the slope of the line. Multiply the slope by 5. What do you notice?

   (d) Find the speed of the car when the mileage is 17.5 mpg.

   (e) Give the slope of the line, with units, and interpret its meaning in the context of the situation.

   (f) Although we can find the m-intercept and nothing about it seems unusual, why should we not interpret its value?

2. The weight w (in grams) of a certain kind of lizard is related to the length l (in centimeters) of the lizard by the equation $w = 22l - 84$. This equation is based on statistical analysis of a bunch of lizards between 12 and 30 cm long.

   (a) Find the weight of a lizard that is 3 cm long. Why is this not reasonable? What is the problem here?

   (b) What is the w-intercept, and why does it have no meaning here?

   (c) What is the slope, with units, and what does it represent?

3. A salesperson earns \$800 per month, plus a 3% commission on all sales. Let P represent the salesperson's gross pay for a month, and let S be the amount of sales they make in a month. (Both are of course in dollars. Remember that to compute 3% of a quantity we multiply by the quantity by 0.03, the decimal equivalent of 3%.)

   (a) Find the pay for the salesperson when they have sales of \$50,000, and when they have sales of \$100,000.

(b) Find the equation for pay as a function of sales, given that this is a linear relationship.

(c) What is the slope of the line, and what does it represent?

(d) What is the P-intercept of the line, and what does it represent?

4. The equation $F = \frac{9}{5}C + 32$ gives the Fahrenheit temperature F corresponding to a given Celsius temperature C. This equation describes a line, with C playing the role of x and F playing the role of y.

   (a) What is the F-intercept of the line, and what does it tell us?

   (b) What is the slope of the line, and what does it tell us?

5. We again consider the manufacture of Widgets by the Acme Company. The costs for one week of producing Widgets is given by the equation $C = 7x + 5000$, where C is the costs, in dollars, and x is the number of Widgets produced in a week. This equation is clearly linear.

   (a) What is the C-intercept of the line, and what does it represent?

   (b) What is the slope of the line, and what does it represent?

   (c) If they make 1,491 Widgets in one week, what is their total cost? What is the cost for each individual Widget made that week? (The answer to this second question should *NOT* be the same as your answer to (b).)

6. The cost y (in dollars) of renting a car for one day and driving x miles is given by the equation $y = 0.24x + 30$. Of course this is the equation of a line. Explain what the slope and y-intercept of the line represent, *in terms of renting the car.*

7. A baby weighs 8 pounds at birth, and four years later the child's weight is 32 pounds. Assume that the childhood weight W (in pounds) is linearly related to age t (in years).

   (a) Give an equation for the weight W in terms of t. *Test it for the two ages that you know the child's weight, to be sure it is correct!*

   (b) What is the slope of the line, with units, and what does it represent?

   (c) What is the W-intercept of the line, with units, and what does it represent?

   (d) Approximately how much did the child weigh at age 3?

8. As the price of an item goes up, the number of units sold goes down, and this relationship is usually linear. If 5000 Widgets are sold at a price of $\$7.50$ and 3500 Widgets are sold at a price of $\$10.00$, determine an equation for the number x of Widgets sold as a linear function of price p. Note that your equation should have the form $x = mp + b$ - test it for the given values to make sure it works!

9. The temperature T, in degrees Fahrenheit, is linearly related to the number x of times that a cricket chirps in a minute. At $60°$F a cricket chirps 90 times a minute, and at $70°$F a cricket chirps 135 times a minute. Find an equation of the form $T = mx + b$ for temperature as a function of the number of chirps. Give the slope and intercept in exact form (fractions, no decimals).

2.5 Other Mathematical Models

Performance Criteria:

2. (j) Solve a problem using a function whose equation is given.
   (k) Determine the equation of a function modeling a situation.
   (l) Give the feasible domain of a function modeling a situation.

Introduction

In this section we will look at what functions have to do with anyone's concept of reality! First we consider this: When we look at a function in the form $y = x^2 - 5x + 2$, we say that "y is a function of x", meaning that y depends on x.

Often we will use letters other than x and y when using functions for "real life" quantities. Consider, for example, the function $h = -16t^2 + 144t$. This equation gives the height h (in feet) of a rock, thrown upward with a starting velocity of 144 feet per second, at any time t seconds after it is thrown. (The $-16t^2$ is the downward effect of gravity. Note that without gravity the equation would simply be $h(t) = 144t$, or "distance equals rate times time.") Here it is logical to use t and h instead of x and y, so we can see from the letter what quantity it represents.

We'll basically be working with two situations:

- Those for which the equation of the function will be provided. The equation will have to be given because it arises from concepts that you are probably not familiar with at this point of your education.
- Those for which you can (and will) create the equation for the function yourself.

It is relatively simple to work with functions whose equations are given, and you have already done this in parts of Chapter 1 and in the previous section. Usually you are given the input and asked for the output,or vice-versa. Since you are already familiar with that, I have not provided examples, but there are some exercises on this.

Finding Equations of Functions

Let's look at finding the equation of a function. As demonstrated in the next example, a good way to do this is to use the given information to find several input-output pairs *BEFORE* trying to find the equation for the function, then checking your equation with some of the input values for which you have already found output values. This process is similar to what we did to find equations for solving problems by guessing and checking. Let's look at an example.

◇ **Example 2.5(a):** A rectangle starts out with a width of 5 inches and a length of 8 inches. At time zero the width starts growing at a constant rate of 2 inches per minute, and the length begins growing by 3 inches per minute. Give an equation for the area A as a function of how many minutes t it has been growing.

Solution: Suppose that the rectangle had been growing for 6 minutes. Then the width would be $5 + 2(6) = 17$ inches and the length would be $8 + 3(6) = 26$ inches. The area would then be $(17)(26) = 442$ square inches. If the rectangle had been growing for 10 minutes, the width would be $5+2(10) = 25$ inches and the length would be $8+3(10) = 38$ inches. The area would then be $(25)(38) = 950$ square inches.

We now suppose that the rectangle had been growing for t minutes, with t unknown. Based on what we have just seen, we know that to get the width of the rectangle we would take the initial width of 5 inches and add on $2t$, which represents how much the width had grown. So the width would be $5 + 2t$. Similarly, the length would be $8 + 3t$. The area is then the length times the width, so we now have

$$A = (5 + 2t)(8 + 3t) = 40 + 15t + 16t + 6t^2 = 6t^2 + 31t + 40.$$

We can show that A is a function of t by writing $A(t) = 6t^2 + 31t + 40$, and we can check this against the results we used to get the formula. For example,

$$A(10) = 6(10)^2 + 31(10) + 40 = 600 + 310 + 40 = 950,$$

which agrees with what we got before we had an equation for the function.

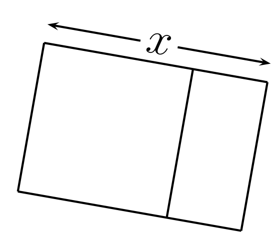
Feasible Domains of Functions

So far we have considered the domain of a function from a mathematical point of view. When using a function to describe a real-world phenomenon, however, we should recognize that not all values of the input variable are reasonable for the situation. Consider the equation $h(t) = -16t^2 + 144t$ describing the height of a projectile above the ground as a function of time after it was launched/thrown. Since the projectile is launched at time zero, it makes no sense to let t be negative. Furthermore, the rock will eventually hit the ground and stop, so the equation is no good for times after that. We'll call the times for which the equation *IS* valid the **feasible domain** of the function. To find when the rock hits the ground we substitute zero for the height and solve:

$$\begin{aligned} 0 &= -16t^2 + 144t \\ 0 &= -16t(t - 9) \\ t = 0 \quad &\text{or} \quad t = 9 \end{aligned}$$

The solution $t = 0$ is telling us that the projectile starts on the ground at time zero, and the solution $t = 9$ is giving the time that the projectile comes back down and hits the ground. The feasible domain of this function is then $[0, 9]$ or $\{t \mid 0 \leq t \leq 9\}$. Note that the feasible domain is far more restrictive than the mathematical domain, which allows all real numbers for t.

◇ **Example 2.5(b):** A farmer is going to create a rectangular field with two "compartments", as shown to the right. He has 2400 feet of fence with which to do this. Letting x represent the dimension shown on the diagram, write an equation for the total area A of the field as a function of x, simplifying your equation as much as possible. Give the feasible domain for the function.



Solution: Lets find the area for a value of x to help us determine an equation for the area as a function of x. Suppose that $x = 500$. Then the opposite side of the field is 500 also, and we have then used $2(500) = 1000$ feet of our fence. That leaves $2400 - 2(500) = 1400$ feet for the remaining *three* sections of fence. That means each of them has length $\frac{1400}{3}$ feet. Since the area of the field is length times width, the area is then

$$A = 500\left(\frac{1400}{3}\right) = 500\left(\frac{2400 - 2(500)}{3}\right)$$

It is not necessary that we actually calculate the area, as we can see now how it is obtained. At this point we can substitute x everywhere for 500 to get

$$A = x\left(\frac{2400 - 2x}{3}\right) = x(800 - \tfrac{2}{3}x) = 800x - \tfrac{2}{3}x^2$$

We now have the equation for the area as a function of x: $A = 800x - \frac{2}{3}x^2$. Now what is the feasible domain of the function? Clearly x must be greater than zero; it can't be negative, and if it is zero there will be no field! Is there an upper limit on x? Well, x certainly can't be larger than 2400, since that is how much fence we have, but we can do better than that. Notice that there are actually two pieces of fence with length x, and *together* they have to be less than 2400. (Again they can't total exactly 2400 because then there would be nothing left for the other three pieces of fence.) Therefore x itself must be less than half of 2400, or less than 1200. In conclusion, the feasible domain is $\{x \mid 0 < x < 1200\}$. This can instead be given using interval notation, as $(0, 1200)$.

Section 2.5 Exercises

To Solutions

1. The height h (in feet) of a rock thrown upward with a starting velocity of 144 feet per second at any time t seconds after it is thrown is given by the equation the function $h = -16t^2 + 144t$. (The $-16t^2$ is the downward effect of gravity. Note that without gravity the equation would simply be $h = 144t$, or "distance equals rate times time.")

   (a) How high will the rock be after 3 seconds?

   (b) How long will it take for the rock to hit the ground? (**Hint:** What is the height of the rock when it hits the ground?)

   (c) Later we will see that the rock will take the same amount of time to reach its high point as it does to come back down. Use this fact to find out *the maximum height the rock will reach*.

   (d) Find the time at which the rock is at a height of 224 feet. You should get two answers - how do you explain this?

2. Since the height of the rock can be found for any time by using the equation, we see that h is a function of t and we could write $h(t) = -16t^2 + 144t$.

   (a) *In words*, tell what $h(4)$ represents, including units. *This DOESN'T mean find* $h(4)$.

   (b) Find a value (or values) of t such that $h(t) = 150$. Give your answer(s) in decimal form, rounded to the nearest hundredth.

   (c) Write a sentence summarizing what you found out in part (b).

   (d) Why does it make no sense to compute $h(11)$? (**Hint:** See your answer to 1(c).)

3. The length L of the skid mark left by a car is proportional to the square of the car's speed s at the start of the skid. This means that there is an equation $L = ks^2$ relating L and s, where k is some constant (fixed number).

   (a) We need to find a value for k. We know that a car traveling 60 miles per hour leaves a 172 foot skid mark. Insert those values and solve for k. Round to the thousandth's place.

   (b) What are the units of k? Note that they must be such that they multiply times the units of s^2 to give the units for L.

   (c) Since k is a constant, it's value doesn't change from what you obtained in (a). Insert the value you obtained to get an equation relating L and s. Use your equation to find out the length of the skid mark that will be left by a car traveling at 50 miles per hour.

   (d) Use your equation to find out how fast the car would be going to leave a skid mark with a length of 100 feet. **Round to the tenth's place**.

4. The intensity I of light at a point is inversely proportional to the square of the point's distance d from the light. This means that I and d are related by an equation of the form $I = \dfrac{k}{d^2}$ for some constant k. The intensity of a light is 180 candlepower at a distance of 12 feet.

   (a) Assuming that a greater candlepower indicates brighter light, should the intensity at 18 feet be less or more than 180 candle power?

   (b) Use a method like that of the previous exercise to determine the intensity of the light at a distance of 18 feet. Does your answer agree with what you speculated in (a)?

5. Clearly, as the price of an item goes up, fewer will be sold, so *the number sold is a function of the price*. An economist for the Acme Company determines that the number w of widgets that can be sold at a price p is given by the equation $w = 20,000 - 100p$.

(a) Find the number of widgets that can be sold at a price of $50, and the number that can be sold at $100. **Show the arithmetic of the units.** Does the number sold increase or decrease, and by how many, when the price is increased from $50 to $100?

(b) Find the amount of increase or decrease when the price is raised from $100 to $150. Tell whether it is an increase or decrease when you give your answer.

(c) At what price would 7,000 widgets be sold?

(d) What is the feasible domain of this function? (**Hint:** Neither the price nor the number of Widgets can be negative.)

6. The amount of money brought in by sales is called the **revenue**. If 47 items that are priced at $10 are sold, then the revenue generated is $470. Consider the widgets from the previous exercise. (You will need to use the function from the previous exercise for most of this exercise!)

   (a) What will the revenue be if the price of a widget is $80? Show/explain how you got your answer.

   (b) What will the revenue be if the price is $110? $140?

   (c) Note that the price increase from $80 to $110 is $30. What is the corresponding increase in revenue? What is the increase in revenue as the price increases another $30, from $110 to $140?

   (d) You should see by now that the revenue depends on the price set for a widget; that is, the *revenue is a function of the price.* Write an equation for this function, and simplify your equation.

   (e) Use your equation to find the revenue if the price of a widget is $80, and check your answer with what you got in part (a). If they don't agree, then your equation is probably incorrect. (It could be that you got part (a) wrong, but that is less likely!)

   (f) What should the price be in order have a revenue of $100,000? Round your answer to the nearest ten cents.

7. Biologists have determined that when x fish of a particular type spawn, the number y of their offspring that survive to maturity is modeled reasonably well by the equation

$$y = \frac{3700x}{x + 400}$$

.

   (a) How many survivors will there be if there are 2500 spawners?

   (b) Your answer to (a) should be rounded to the nearest whole number. Why?

(c) How many spawners are needed to get 3000 survivors?

(d) How many spawners must there be so that the number of survivors equal the number of spawners?

8. If a person takes a 30 milligram dose of a particular medicine, the amount of medicine left in the person's body at t hours after the medicine was taken is given by the equation $A(t) = 30(0.7)^t$. This is an example of an **exponential function**; we'll study them in more detail in Chapter 6. You may find it less confusing to work with the equation in the form $A = 30(0.7)^t$. Find the amount of medicine left in a person's body 3 hours after taking the medicine. Round to the tenth's place and give units with your answer. Remember that you must apply the exponent to the 0.7 *before* you multiply by 30, because of the order of operations.

9. Two people, who are 224 feet apart, start walking toward each other at the same time. One is walking at a rate of 1.5 feet per second and the other at 2 feet per second. Let time zero be when they start walking toward each other; clearly their distance apart is a function of the time elapsed since time zero.

   (a) Find out how far apart the people are at times $t = 0$, 1, 3 and 8 seconds. You may wish to record your results in a small table of two columns, the first labeled t and the second labeled as d.

   (b) Write a brief explanation, *in words*, of how to find the distance d for any time t.

   (c) Write an equation for the distance d as a function of the time t. *Give units with any constants in your equation.* Test it by finding $d(0)$, $d(3)$ and $d(8)$, *showing the arithmetic of the units as well as the numbers*, and making sure the results agree with what you got in (a).

   (d) Use your equation from (c) to determine how far apart the people will be after 20 seconds. *try showing how the units of the constants are worked with in this case.*

   (e) When will the people be 130 feet apart? When will they meet?

   (f) What is the mathematical domain of the function? What is the feasible domain?

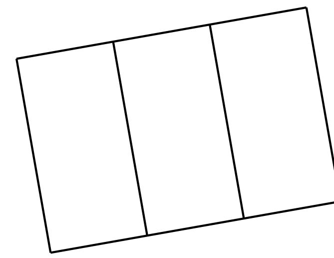
10. Recall that for a right triangle with legs of lengths a and b and hypotenuse with length c the Pythagorean Theorem tells us that $a^2 + b^2 = c^2$. Suppose that the legs of a right triangle start out with lengths 3 feet and 5 feet, and they both increase at 4 feet per hour.

    (a) Find the perimeter for several times.

    (b) Find, *but don't simplify*, a formula for the perimeter P of the triangle as a function of time t.

    (c) Simplify your formula from part (b).

11. A farmer is going to create a rectangular field with three "compartments", as shown to the right. He has 1000 feet of fence with which to do this. Letting x represent the dimension shown on the diagram, write an equation for the total area A of the field as a function of x. Simplify your equation as much as possible. **HINT:** To find your equation, experiment with a few specific numerical values of x, like you did (with t) in the previous exercise.



12. What is the mathematical domain of the function from the previous exercise? What is the feasible domain?

13. Billy and Bobby have a lawn service (called Billybob's Lawn Care). Billy, the faster of the two, can mow a three lawns every two hours by himself, and Bobby can only mow two lawns in two hours by himself.

    (a) How many lawns can each boy mow in one hour by themselves?

    (b) Find an equation for the number L of lawns that the two can mow in t hours *when working together*.

    (c) How long does it take the two of them together to mow one lawn? Give your answer in decimal form, remembering that it is in hours. Then give the answer in minutes.

    (d) What is the mathematical domain of the function? What is the feasible domain?

14. A 10 foot ladder is leaned against the side of a building. Note that the base of the ladder can be put at different distances from the wall, but changing the distance between the base of the ladder and the wall will cause the height of the top of the ladder to change.

    (a) There are three lengths or distances of interest here. What are they? Are any of them variables and, if so, which ones? Are any of them constants and, if so, which ones?

    (b) As the distance of the base of the ladder from the wall decreases, what happens to the top of the ladder?

    (c) Clearly the height to the top of the ladder is related to the distance of the base of the ladder from the wall. Write an equation relating these two variables, using h for the height and d for the distance of the base of the ladder from the wall.

    (d) Use your equation from (c) to determine the height of the ladder when the base is 6 feet from the wall. Then use your equation to determine the height of the ladder when the base is 2 feet from the wall.

    (e) You second answer to (d) should not be a whole number. Give it as a decimal, rounded to the nearest tenth of a foot. Then give it in simplified square root form.

    (f) Give the advantages and disadvantages of each of the two forms that you gave for your answer to (e).

(g) Give your answer in inches, rounded to the nearest whole inch. Then give it in feet and inches.

15. Consider the situation from the previous exercise.

    (a) Note that you can solve your equation from (c) to get $h = \pm\sqrt{100 - d^2}$, where h is the height of the top of the ladder and d is the distance from the bottom of the ladder to the base of the wall. Why can we disregard the negative sign?

    (b) Find the feasible domain for the function $h = \sqrt{100 - d^2}$.

16. The width of a rectangle is one-third its length.

    (a) Find the area of the rectangle as a function of the length of the rectangle. This means you should have an equation of the form A equals some mathematical expression containing l.

    (b) Find the area as a function of the width.

    (c) Find the perimeter P as a function of the length, then as a function of the width. (Remember that perimeter is the distance around the figure.)

17. The Acme Company has a factory that produces Widgets. The company spends \$5000 per week operating their factory (mortgage payments, heating, etc.). Additionally, it takes \$7 in parts, energy and labor to produce a Widget.

    (a) What are Acme's total costs for a week in which 8,000 Widgets are produced? What are the costs for a week in which 12,000 Widgets are produced?

    (b) Clearly the costs for a week are a function of the number of Widgets produced. Let C represent the costs for a week and let x represent the number of Widgets produced that week. Write an equation for the costs as a function of the number of Widgets produced.

18. Suppose that the price of gasoline is \$2.70 per gallon, and you are going to put some number g gallons of gas in your tank.

    (a) Write an equation for the cost C (in dollars) to you as a function g. gallons of gas in your car.

    (b) What does the feasible domain of this function depend on? Be as specific as possible!

2.6 A Return to Graphing

Performance Criteria:

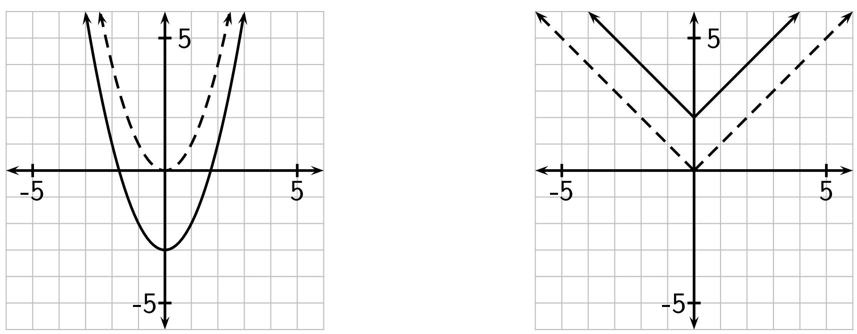
5. (m) Given the graph of a function, sketch or identify translations of the function.

   (n) Given the graph of a function, sketch or identify vertical reflections of the function.

Let's begin with an example.

◇ **Example 2.6(a):** Sketch the graph of $y = x^2$, dashed. On the same grid, graph $f(x) = x^2 - 3$, solid. On a different grid, repeat for $y = |x|$ and $g(x) = |x| + 2$.

Solution: We'll skip the details of finding points to plot, which was covered thoroughly in Chapter 1. The graphs are shown below.



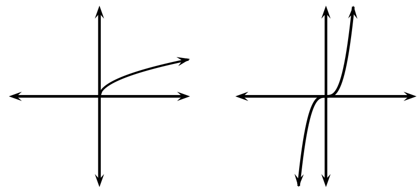
$y = x^2$ and $f(x) = x^2 - 3$ $y = |x|$ and $g(x) = |x| + 2$

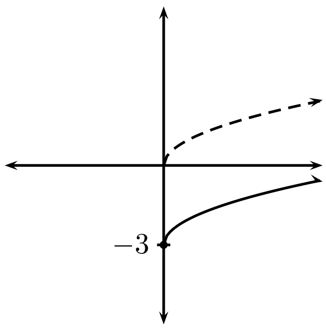
The point of this example, of course, was not simply to graph the given functions. We can observe the effect of adding or subtracting a constant on the graph of a function. We see that adding two to $y = |x|$ causes the graph to move up two units, and subtracting four from $y = x^2$ moves the graph down by four units. We call any movement that does not change the shape or orientation of a graph a **translation** of of the graph. So we are seeing that adding or subtracting a constant to a function translates (or "shifts") the graph up or down by the absolute value of the constant, with the shift being upward if the constant is positive and downward if the constant is negative.

Taking advantage of this fact, and others we'll see in this section, requires knowing the graphs of some common functions. At this point the reader should review the graphs of some basic equations (which are of course functions) in Section 1.7. Here is another example demonstrating what we discovered above.

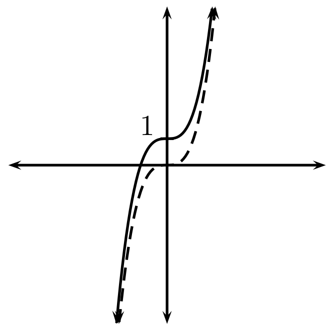
◇ **Example 2.6(b):** Sketch the graphs of $y = \sqrt{x} - 3$ and $y = x^3 + 1$.

Solution: The graphs of $y = \sqrt{x}$ and $y = x^3$ are shown to the right. To obtain the graph of $y = \sqrt{x} - 3$ we simply shift the graph of $y = \sqrt{x}$ down by three units, and we obtain the graph of $y = x^3 + 1$ we shift the graph of $y = x^3$ up by one unit. The resulting graphs are shown at the top of the next page.





$y = \sqrt{x}$ (dashed) and
$y = \sqrt{x} - 3$ (solid)

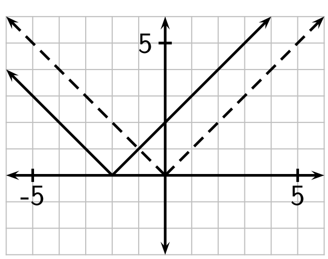


$y = x^3$ (dashed) and
$y = x^3 + 1$ (solid)

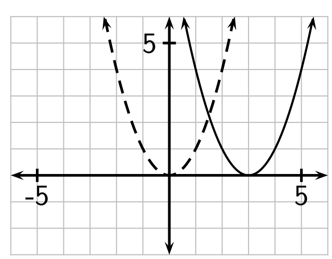
For the next example, compare the functions being graphed with those graphed in Example 2.6(a), and observe carefully the difference between the functions and the resulting graphs.

◇ **Example 2.6(c):** Sketch the graph of $y = |x|$, dashed. On the same grid, graph $g(x) = |x + 2|$, solid. On a different grid, repeat for $y = x^2$ and $h(x) = (x - 3)^2$.

Solution: We recall the idea that for the function g the value $x = -2$ is important because it makes $x + 2$ zero. If we evaluate g for some values of x on either side of -2 and plot the resulting points and graph, we get the solid graph below and to the left. Similarly, the critical value of x for the function h is $x = 3$ because it makes $x - 3$ zero. Evaluating for values near $x = 3$ and plotting results in the graph to the right below.

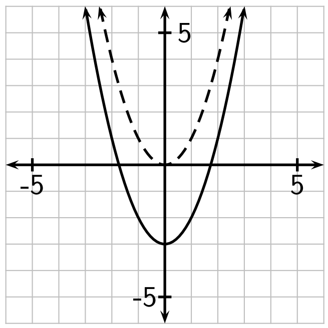


$y = |x|$ and $g(x) = |x + 2|$

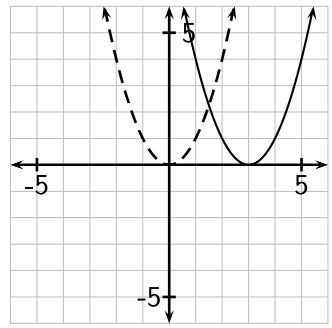


$y = x^2$ and $h(x) = (x - 3)^2$

What can we gather in general from these examples? Let's compare the second graph above with the one from Example 2.6(a):



$y = x^2$ and $f(x) = x^2 - 3$



$y = x^2$ and $h(x) = (x-3)^2$

In both cases we are modifying $y = x^2$ by subtracting three, but the difference is this: In $f(x) = x^2 - 3$ we are subtracting three *after* squaring, and in $h(x) = (x-3)^2$ we are subtracting three *before* squaring. In the first case we simply obtain y values for $y = x^2$ and subtract three from all of them, causing the graph to shift downward by three units. In the second case, evaluating $h(x) = (x-3)^2$ for $x = 3$ gives the same y value as evaluating $y = x^2$ at $x = 0$. In fact, evaluating $h(x) = (x-3)^2$ for any x value gives the same result as evaluating $y = x^2$ for a value of x that is three less. This may be a bit confusing, but the net result is that *subtracting three before squaring results in shifting the graph to the right by three*. Let's summarize all of this:

Translations (Shifts) of Graphs:

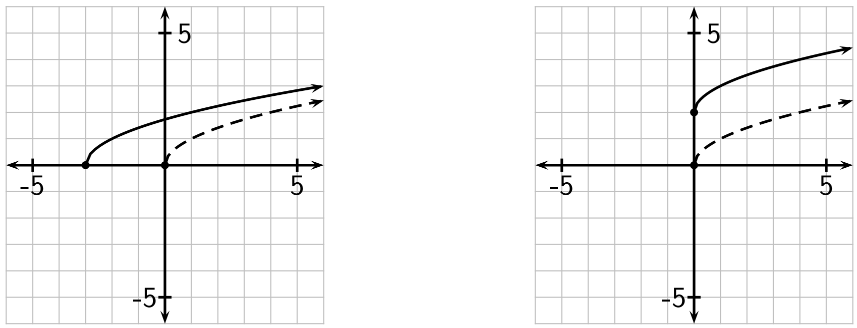
Let a and b be *positive* constants and let $f(x)$ be a function.

- The graph of $y = f(x+a)$ is the graph of $y = f(x)$ shifted a units to the left. That is, what happened at zero for $y = f(x)$ happens at $x = -a$ for $y = f(x+a)$.
- The graph of $y = f(x-a)$ is the graph of $y = f(x)$ shifted a units to the right. That is, what happened at zero for $y = f(x)$ happens at $x = a$ for $y = f(x-a)$.
- The graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up by b units.
- The graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down by b units.

Note that when we add or subtract the value a *before* the function acts, the shift is in the x direction, and the direction of the shift is the opposite of what our intuition tells us should happen. When we add or subtract b *after* the function acts, the shift is in the y direction and the direction of the shift is up for positive and down for negative, as our intuition tells us it should be.

◇ **Example 2.6(d):** Sketch the graphs of $y = \sqrt{x+3}$ and $y = \sqrt{x} + 2$.

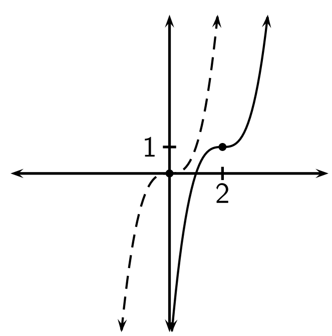
Solution: The graph of $y = \sqrt{x}$ is shown as the dashed curve on both grids below. For $y = \sqrt{x+3}$ we are adding three *before* taking the root, so the graph is shifted three units to the left, as shown on the left grid below. For $y = \sqrt{x} + 2$, two is added *after* taking the square root, so the graph is shifted up by two units. This is shown on the grid to the right below.



$y = \sqrt{x+3}$ (solid) $\qquad$ $y = \sqrt{x} + 2$ (solid)

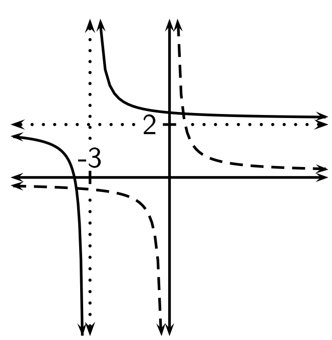
◇ **Example 2.6(e):** Sketch the graph of $y = (x-2)^3 + 1$ on the same grid with the graph of $y = x^3$.

Solution: We first note that this is a modification of $y = x^3$, whose graph is the dashed curve to the right. Note that we are modifying by first subtracting two *before* cubing and adding one *after* cubing. Thus the graph is moved two units to the right and one unit up. The final result is seen in the solid graph to the right.



◇ **Example 2.6(f):** Sketch the graph of $y = \dfrac{1}{x+3} + 2$ on the same grid with the graph of $y = \dfrac{1}{x}$.

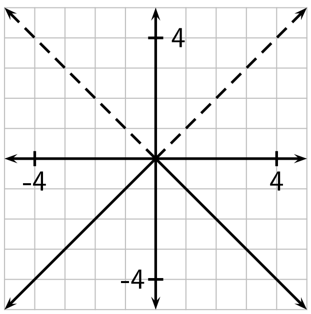
Solution: This is a modification of $y = \dfrac{1}{x}$, shown as the dashed curve to the right. In this case we are modifying by adding three before the "one over," and adding two afterward. The resulting graph is moved three units to the left and two units up. The final result is seen in the solid graph to the right.



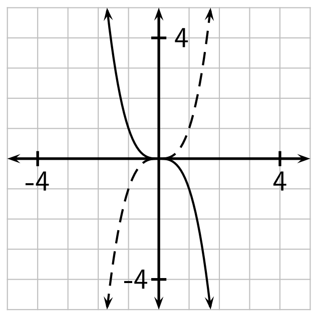
The vertical and horizontal dotted lines in the above graph *ARE NOT* parts of the graph of the equation $y = \dfrac{1}{x+3} + 2$, but they help us see where the graph (the solid curve) is located. The vertical, line, whose equation is $x = -3$, is called a **vertical asymptote** of the graph, and the horizontal line $y = 2$ is called a **horizontal asymptote**.

Vertical Reflections (Flips) of Graphs

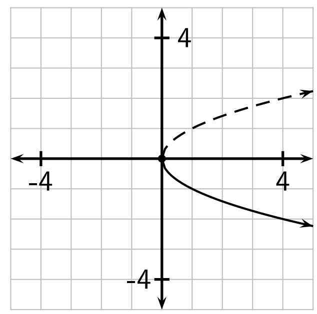
On the three grids below we see the graphs of $y = -|x|$, $y = -x^3$ and $y = -\sqrt{x}$ (solid), along with $y = |x|$, $y = x^3$ and $y = \sqrt{x}$ (dashed).



$y = |x|$ and $y = -|x|$



$y = x^3$ and $y = -x^3$

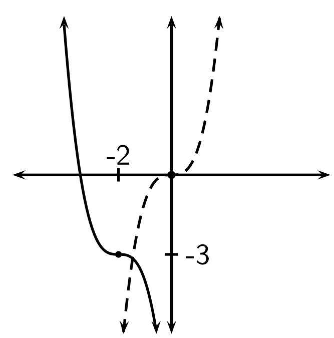


$y = \sqrt{x}$ and $y = -\sqrt{x}$

It may not be readily apparent at first glance what is happening here, but the effect of multiplying a function by negative one (which is what is really happening when we make the function negative) is to flip its graph vertically over the x-axis. We call this a **reflection** of the graph over the x-axis. We will often see such reflections combined with the previously seen translations.

◇ **Example 2.6(g):** Sketch the graph of $y = -(x+2)^3 - 3$ on the same grid with the graph of $y = x^3$.

Solution: The graph of $y = x^3$ is the dashed curve to the right. The effect of adding two before cubing is to shift the graph to the left by two units. The multiplication by negative one takes place *after* the cubing (order of operations!) and results in the graph being flipped vertically over the x-axis. Finally, three is subtracted at the end, shifting the graph down by three units. The final result is the solid graph to the right.



Section 2.6 Exercises

To Solutions

You will check some of your answers for this section by graphing your results on either a graphing calculator or using an online grapher. I would again suggest

www.desmos.com,

which allows you to graph two functions on the same graph, as we have been doing in this section.

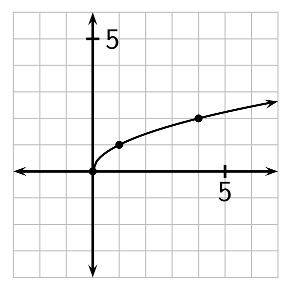
1. For each of the quadratic functions below,

   - sketch the graph of $y = x^2$ **accurately** on a coordinate grid,
   - **on the same grid**, sketch what you think the graph of the given function would look like,
   - graph $y = x^2$ and the given function together on your calculator or Desmos to check your answer

   (a) $f(x) = (x+1)^2$ (b) $y = x^2 - 4$

   (c) $g(x) = (x+2)^2 - 1$ (d) $h(x) = -(x-4)^2 + 2$

2. The graph of $y = \sqrt{x}$ is shown to the right, with the three points $(0,0)$, $(1,1)$ and $(4,2)$ indicated with dots. For each of the following, sketch what you think the graph of each function would look like, along with a dashed copy of the graph of $y = \sqrt{x}$. On the graph of the new function, indicate three points showing the new positions of the three previously mentioned points. Check your answers in the back, making sure to see that you have the correct points plotted.



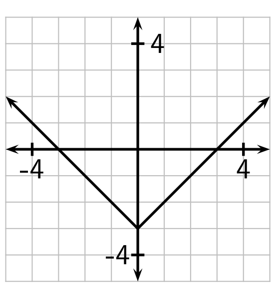
$y = \sqrt{x}$

   (a) $y = \sqrt{x-1}$ (b) $f(x) = \sqrt{x} - 3$

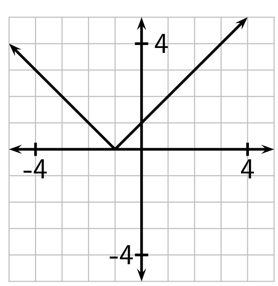
   (c) $g(x) = -\sqrt{x-2} + 4$

3. For each of the graphs shown, give a function of the form $y = \pm|x \pm a| \pm b$ whose graph would be the one shown. Check your answers by graphing them with some sort of technology and seeing if the results are as shown.

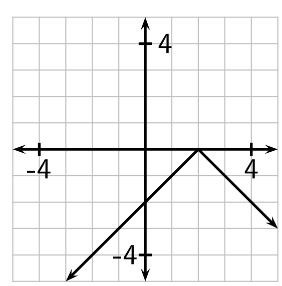
(a)

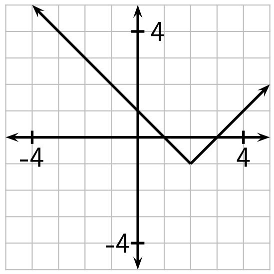
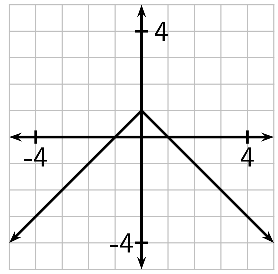
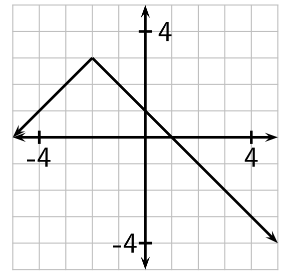


(b)



(c)



(d)  (e)  (f) 

4. For each of the following, graph $y = x^3$ and the given function on the same graph. Check your answers in the solutions.

   (a) $f(x) = -x^3$ (b) $y = x^3 + 4$ (c) $g(x) = (x-1)^3$

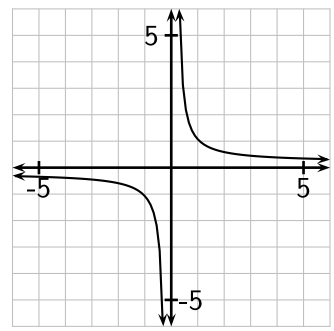
   (d) $y = (x+3)^3 + 1$ (e) $h(x) = -(x+2)^3$

5. For each of the following, graph $y = \frac{1}{x}$ (its graph is shown below and to the right) and the given function on the same graph, with the graph of $y = \frac{1}{x}$ dashed.

   (a) $y = \dfrac{1}{x+3}$ (b) $y = \dfrac{1}{x} + 3$

   (c) $y = -\dfrac{1}{x}$ (d) $y = \dfrac{1}{x+2} + 1$

   (e) $y = -\dfrac{1}{x-2}$



$y = \dfrac{1}{x}$

6. Give the domain and range of each function without graphing them (you may wish to think about what their graphs would look like, though).

   (a) $y = (x+3)^2 - 2$ (b) $f(x) = \sqrt{x+3}$ (c) $g(x) = -x^2 + 5$

   (d) $h(x) = \dfrac{1}{x-2}$ (e) $y = -|x+4| + 7$ (f) $y = \sqrt{x-2} + 1$

2.7 Chapter 2 Exercises

To Solutions

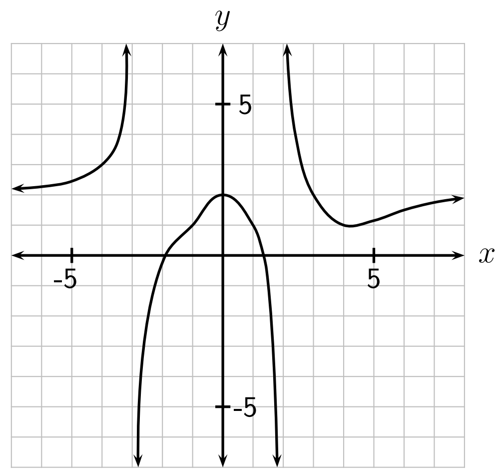
1. For each of the following,

   - attempt to determine the domain of the function algebraically
   - give the domain using either interval notation or set builder notation
   - Graph the function using technology, and ee if the graph seems to support your answer
   - check your answer in the back of the book

   (a) $f(x) = \dfrac{x+3}{x^2-x}$ (b) $h(x) = \sqrt{x+2}$ (c) $g(x) = \dfrac{8}{x^2-1}$

   (d) $f(x) = \dfrac{x+2}{x^2-5x+4}$ (e) $g(x) = \sqrt{3x-2}$ (f) $y = \dfrac{x^2}{x^2+3}$

2. The graph of a function h is shown below and to the right. The left and right tails continue outward with y values getting closer and closer to, but never reaching, 2. The tails going up and down continue that way, getting ever closer to, but never touching, the vertical lines $x = -3$ and $x = 2$.



   (a) Give the domain of the function.

   (b) Give the range of the function.

   (c) Give the values of $h(-4)$ and $h(4)$.

   (d) Give *all* values of x for which $h(x) = 1$.

   (e) Give *all* values of x for which $h(x) = 2$.

   (f) Give the intervals on which h is decreasing.

   (g) Give the intervals where h is negative.

   (h) Does h have any relative maxima or minima? If so, give the values of x where they are, and give the function value at each of those x values.

   (i) Repeat (h) for absolute maxima and minima.

3 Quadratic Functions, Circles, Average Rate of Change

Outcome/Performance Criteria:

3. Understand and apply quadratic functions. Find distances and midpoints, graph circles. Solve systems of two non-linear equations, find average rates of change and difference quotients.

   (a) Find the vertex and intercepts of a quadratic function; graph a quadratic function given in standard form or factored form.

   (b) Change a quadratic function from standard form to factored form, and vice-versa. Change from vertex form to standard form.

   (c) Graph a parabola whose equation is given in $y = a(x-h)^2+k$ form, relative to $y = x^2$.

   (d) Given the vertex of a parabola and one other point on the parabola, give the equation of the parabola.

   (e) Put the equation of a parabola in standard form $y = a(x-h)^2 + k$ by completing the square.

   (f) Solve a linear inequality; solve a quadratic inequality.

   (g) Solve a problem using a given quadratic model; create a quadratic model for a given situation.

   (h) Find the distance between two points in the plane; find the midpoint of a segment between two points in a plane.

   (i) Graph a circle, given its equation in standard form. Given the center and radius of a circle, give its equation in standard form.

   (j) Put the equation of a circle in standard form

   $$(x-h)^2 + (y-k)^2 = r^2$$

   by completing the square.

   (k) Solve a system of two non-linear equations.

   (l) Find the average rate of change of a function over an interval, from either the equation of the function or the graph of the function. Include units when appropriate.

   (m) Find and simplify a difference quotient.

3.1 Quadratic Functions

Performance Criteria:

3. (a) Find the vertex and intercepts of a quadratic function; graph a quadratic function given in standard form or factored form.

   (b) Change a quadratic function from standard form to factored form, and vice-versa. Change from vertex form to standard form.

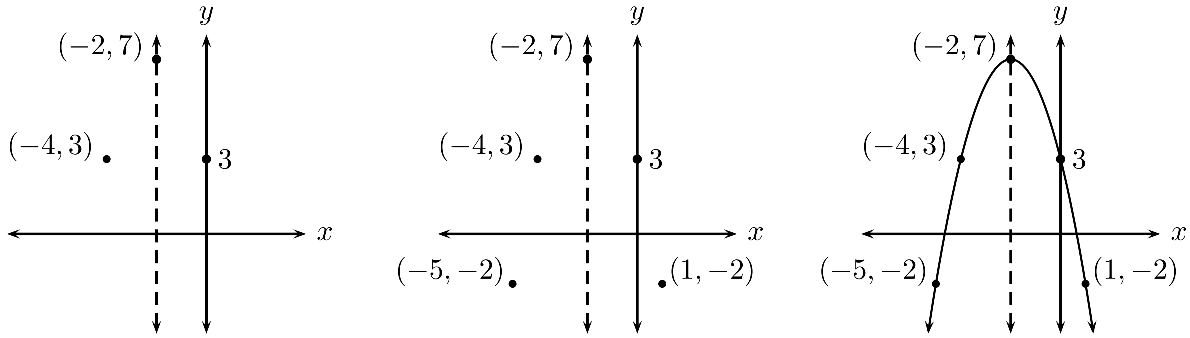
A **quadratic function** is a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. You have seen a number of these functions already, of course, including several in "real world" situations $(h = -16t^2 + v_0t + h_0,$ for example). We will refer to the value a, that x^2 is multiplied by, as the **lead coefficient**, and c will be called the **constant term**. We already saw and graphed such functions in Section 1.7, as equations of the form $y = ax^2 + bx + c$. Let's recall what we know about the graphs of such functions:

- The graph of $f(x) = ax^2 + bx + c$ is a parabola. The parabola opens up if $a > 0$ and down if $a < 0$.
- Every quadratic function has either an absolute maximum or an absolute minimum, depending on whether it opens down or up. The absolute maximum or minimum occurs at the x-coordinate of the vertex, and its value is the y-coordinate of the vertex.
- The vertex of the parabola has x-coordinate $x = -\dfrac{b}{2a}$ and the y-coordinate of the vertex is obtained by evaluating the function for that value of x. The vertical line passing through the vertex (so with equation $x = -\frac{b}{2a}$) is called the **axis of symmetry** of the parabola.
- The y-intercept of the parabola is $y = c$.
- Other than the vertex, every point on a parabola corresponds to another point on the other side of the parabola that has the same y-coordinate and is equidistant from the axis of symmetry.

Let's review Example 1.7(b) before continuing.

◇ **Example 1.7(b):** Graph the equation $y = -x^2 - 4x + 3$. Indicate clearly and accurately five points on the parabola.

Solution: First we note that the graph will be a parabola opening downward, with y-intercept 3. (This is obtained, *as always*, by letting $x = 0$.) The x-coordinate of the vertex is $x = -\dfrac{-4}{2(-1)} = -2$. Evaluating the equation for this value of x gives us $y = -(-2)^2 - 4(-2) + 5 = -4 + 8 + 3 = 7$, so the vertex is $(-2, 7)$. On the first graph at the top of the next page we plot the vertex, y-intercept, the axis of symmetry, and the point opposite the y-intercept, $(-4,3)$. To get two more points we can evaluate the equation for $x=1$ to get $y=-1^2-4(1)+3=-2$. We then plot the resulting point $(1,-2)$ and the point opposite it, as done on the second graph. The third graph then shows the graph of the function with our five points clearly and accurately plotted.



We will now be working with three different forms of quadratic functions:

- **Standard Form:** This is the form discussed above, that you are already familiar with.
- **Factored Form:** The factored form of a quadratic function is $f(x)=a(x-x_1)(x-x_2)$, where a is the same as the a of the standard form. x_1 and x_2 are just numbers.
- **Vertex Form:** The vertex form of a quadratic function is $f(x)=a(x-h)^2+k$, with a again being the same as for the other two forms. h and k are just numbers.

We'll explore all three forms just a bit in this section, and the next section will be devoted exclusively to the vertex form. First let's see how to change from factored or vertex form to standard form, which entails a bit of basic simplification.

◇ **Example 3.1(a):** Put $f(x)=-3(x+1)(x+4)$ and $h(x)=\frac{1}{3}(x+3)^2-2$ into standard form.

Solution: For quadratic functions like f that are in factored form, it usually easiest to multiply the two linear factors $x+1$ and $x+4$ first, then multiply in the constant -3:

$$f(x)=-3(x+1)(x+4)=-3(x^2+5x+4)=-3x^2-15x-12$$

The function h is given in vertex form. In this case we must follow the order of operations, first squaring $x+3$, then multiplying by $\frac{1}{3}$ and, finally, subtracting two:

$$h(x)=\tfrac{1}{3}(x+3)^2-2=\tfrac{1}{3}(x^2+6x+9)-2=(\tfrac{1}{3}x^2+2x+3)-2=\tfrac{1}{3}x^2+2x+1$$

The standard forms of the functions are $f(x)=-3x^2-15x-12$ and $h(x)=\frac{1}{3}x^2+2x+1$.

Next we see how to change from standard form to factored form.

◇ **Example 3.1(b):** Put $g(x) = -2x^2 + 8x + 10$ into factored form.

Solution: Here we first need to factor -2 out of the quadratic expression $-2x^2+8x+10$. To do this we must determine values e and f for which

$$-2(x^2 + ex + f) = -2x^2 + 8x + 10$$

We can see that e must be -4 and $f = -5$. We then have

$$g(x) = -2x^2 + 8x + 10 = -2(x^2 - 4x - 5) = -2(x + 1)(x - 5),$$

where the last expression was obtained by factoring $x^2 - 4x - 5$. Thus the factored form of $g(x) = -2x^2 + 8x + 10$ is $g(x) = -2(x + 1)(x - 5)$.

◇ **Example 3.1(c):** Put $h(x) = \frac{1}{4}x^2 + \frac{1}{2}x - 2$ into factored form.

Solution: Here we first need to factor $\frac{1}{4}$ out of $\frac{1}{4}x^2 + \frac{1}{2}x - 2$. This is most easily done by first getting a common denominator of four in each coefficient, then factoring the four out of the bottom of each coefficient as $\frac{1}{4}$:

$$h(x) = \tfrac{1}{4}x^2 + \tfrac{1}{2}x - 2 = \tfrac{1}{4}x^2 + \tfrac{2}{4}x - \tfrac{8}{4} = \tfrac{1}{4}(x^2 + 2x - 8)$$

We can then factor $x^2 + 2x - 8$ to get the factored form $h(x) = \frac{1}{4}(x + 4)(x - 2)$.

The factored form of a quadratic function is $f(x) = a(x - x_1)(x - x_2)$. In the above example, $a = \frac{1}{4}$, $x_1 = 4$ and $x_2 = -2$. We will now see that it is fairly easy to graph a quadratic function that is given in factored form.

◇ **Example 3.1(d):** Graph the function $h(x) = \frac{1}{4}(x + 4)(x - 2)$. Plot the intercepts, vertex and one more point.

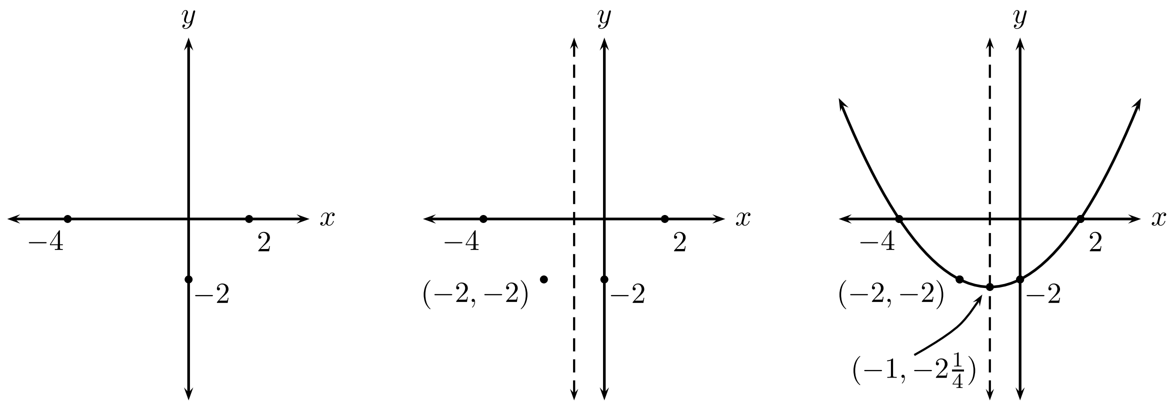
Solution: We first note that $x = -4$ and $x = 2$ are important values of x. Why? Well, it is easily seen that $h(-4) = h(2) = 0$ so, because $h(x)$ can be thought of as y, $y = 0$ when $x = -4$ or $x = 2$. This gives us the two x-intercepts, which are plotted on the axes to the left on the next page. Letting $x = 0$ gives us the y-intercept $y = g(0) = \frac{1}{4}(4)(-2) = -2$, also plotted to the left on the next page.

Using the fact that pairs of points on either side of a parabola are equidistant from the axis of symmetry, we can see that the axis of symmetry is the vertical line halfway between $x = -4$ and $x = 2$. The easiest way to find a value halfway between two numbers is to average them (you will see this idea gain soon!), so the axis of symmetry is $x = \frac{-4+2}{2} = -1$. This is shown on the axes in the center at the top of the next page, along with an additional point with the same y-coordinate as the y-intercept but equidistant from the axis of symmetry on the other side.

To finish up our graph we should at least find the vertex, and maybe a couple more points if asked. The x-coordinate of the vertex is -1, so its y-coordinate is

$$h(-1) = \tfrac{1}{4}(-1 + 4)(-1 - 2) = -\tfrac{9}{4} = -2\tfrac{1}{4}.$$

(Note that we DO NOT have to put the function in standard form to evaluate it.) The vertex is then at $\left(-1, -2\frac{1}{4}\right)$. This, and the other points we've found are plotted on the third set of axes below, and the graph of the function is also shown on that grid.



Section 3.1 Exercises To Solutions

1. Put each of the following quadratic functions in standard form.

   (a) $f(x) = -\frac{1}{5}(x-2)(x+3)$ (b) $g(x) = 3(x-1)^2 + 2$

   (c) $y = -2(x-3)(x+3)$ (d) $h(x) = \frac{1}{2}(x+2)^2 - 5$

2. Put each of the following functions in factored form.

   (a) $f(x) = -\frac{1}{4}x^2 + x + \frac{5}{4}$ (b) $y = 2x^2 - 8x + 6$

   (c) $g(x) = \frac{1}{2}x^2 + 5x + \frac{9}{2}$ (d) $h(x) = -5x^2 + 5$

   (e) $y = 3x^2 - 15x + 18$ (f) $f(x) = \frac{1}{4}x^2 + \frac{3}{2}x - 4$

   (g) $y = -\frac{1}{3}x^2 + \frac{1}{3}x + 2$ (h) $g(x) = -x^2 + 5x + 14$

   Hint for part (h): Factor -1 out.

3. A function $f(x) = ax^2 + bx + c$ is quadratic as long as a is not zero, but either of b or c (or both) *CAN* be zero. Put each of the following in factored form by first factoring out any common factor, then factoring what remains, if possible.

   (a) $y = 5x^2 - 5$ (b) $h(x) = x^2 - 4x$

4. Do the following for each quadratic function below.

   - Determine the x- and y- intercepts.
   - Determine the coordinates of the vertex in the manner described in Example 3.1(d).
   - Plot the intercepts and vertex, and then one more point that is the "partner" to the y-intercept, and sketch the graph of the function.

   Check your answers to the first two items in the back of the book, and check your graphs by graphing the function on your calculator or a computer/tablet/phone.

   (a) $f(x) = -\frac{1}{4}(x+1)(x-5)$
   (b) $y = 2(x-3)(x-1)$
   (c) $g(x) = \frac{1}{2}(x+1)(x+9)$
   (d) $h(x) = -5(x+1)(x-1)$
   (e) $y = -(x+2)(x-7)$
   (f) $f(x) = \frac{1}{4}(x+8)(x-2)$

5. Do the following for the function $y = -(x+1)^2 + 16$:

   (a) Put the equation of the function into standard form.
   (b) Put the equation of the function into factored form.
   (c) Give the coordinates of the vertex of the parabola.
   (d) Graph the parabola, checking your answer with your calculator or a computer/tablet/phone.

6. Repeat Exercise 5 for the function $f(x) = \frac{1}{3}(x-2)^2 - \frac{1}{3}$.

7. Find the equation in standard form of the parabola with x-intercepts -1 and 3 and y-intercept 1, using the following suggestions:

   - Use the x-intercepts to find the equation in factored form $y = a(x-x_1)(x-x_2)$ first - you will not know a at this point.
   - What is the x-coordinate for the y-intercept? Put the coordinates of the y-intercept in for x and y and solve to find a.
   - Multiply out to get the standard form of the equation of the parabola.
   - Check your answer by graphing it with your calculator or a computer/tablet/phone and seeing if it in fact has the given intercepts.

8. Find the equation, in standard form, of the parabola with one x-intercept of -2 and vertex $(1, -6)$. (**Hints:** What is the other x-intercept? Now do something like you did for the previous exercise.) Again, check your answer by graphing it on your calculator or a computer/tablet/phone.

9. Discuss the x-intercept situation for quadratic functions. (Do they always have at least one x-intercept? Can they have more than one?) Sketch some parabolas that are arranged in different ways (but always opening up or down) relative to the coordinate axes to help you with this.

10. Discuss the existence of relative and absolute maxima and/or minima for quadratic functions.

11. Consider the function $f(x) = (x-3)^2 - 4$.

    (a) Use the ideas of Section 2.6 to graph the function along with a dashed version of $y = x^2$.

    (b) Put the equation in standard form.

    (c) Use $x = -\dfrac{b}{2a}$ with the standard form of the equation to find the coordinates of the vertex of the parabola. If the results don't agree with your graph, figure out what is wrong and fix it.

    (d) Graph the standard form equation on your calculator or computer/tablet/phone and see if it agrees with your graph from (a). If it doesn't, figure out what is wrong and fix it.

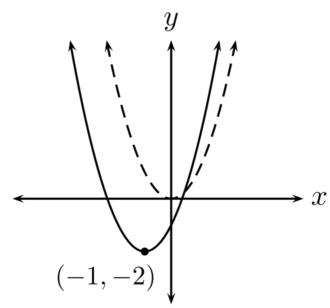
3.2 Quadratic Functions in Vertex Form

Outcome/Criteria:

3. (c) Graph a parabola with equation given in $y = a(x-h)^2 + k$ form, relative to $y = x^2$.
   (d) Given the vertex of a parabola and one other point on the parabola, give the equation of the parabola.
   (e) Put the equation of a parabola in the form $y = a(x-h)^2 + k$ by completing the square.

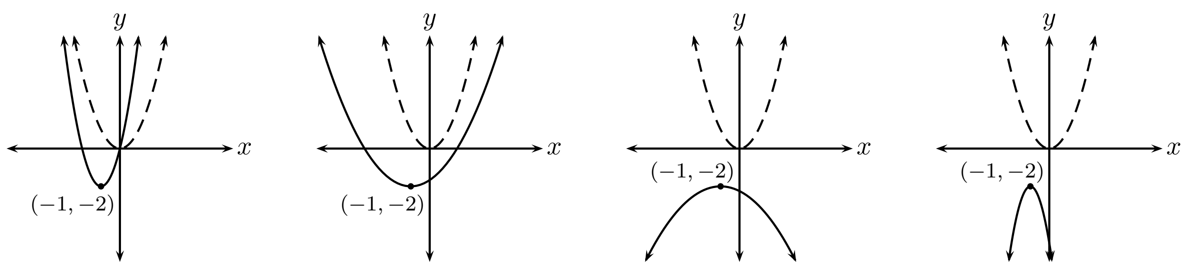
Graphing $y = a(x-h)^2 + k$

From Section 2.6 we know that the graph of the function $f(x) = (x+1)^2 - 2$ is the graph of $y = x^2$ shifted one unit left and two units down. This is shown to the right, where the dashed curve is the graph of $y = x^2$ and the solid curve is the graph of $f(x) = (x+1)^2 - 2$. Suppose now that we wish to graph an equation of the form $f(x) = a(x+1)^2 - 2$ for some value of a not equal to one. The question is "How does the value of a affect the graph?" The following example gives us a start on answering that question.



◇ **Example 3.2(a):** Graph $f(x) = a(x+1)^2 - 2$ for $a = 2, \frac{1}{3}, -\frac{1}{4}$ and -3.

Solution: Below are the respective graphs as solid curves, with $y = x^2$ also plotted on each grid as a dashed line.



$f(x) = 2(x+1)^2 - 2$ $\quad f(x) = \frac{1}{3}(x+1)^2 - 2$ $\quad f(x) = -\frac{1}{4}(x+1)^2 - 2$ $\quad f(x) = -3(x+1)^2 - 2$

From the above example we first observe that when a is positive, the graph of $f(x) = a(x+1)^2 - 2$ opens upward, and when a is negative the graph opens downward. Less obvios is the fact that when the absolute value of a is greater than one the graph of $f(x) = a(x+1)^2 - 2$ is narrower than the graph of $y = x^2$, and when the absolute value of a is less than one the graph of $f(x) = a(x+1)^2 - 2$ is wider than the graph of $y = x^2$. We can summarize all this, as well as some of what we already knew, in the following.

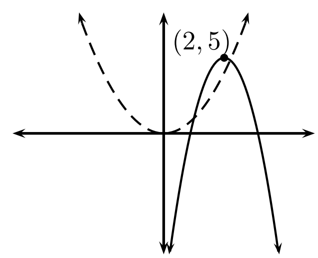
The Equation $y = a(x-h)^2 + k$

The graph of $y = a(x-h)^2 + k$ is a parabola. The x-coordinate of the vertex of the parabola is the value of x that makes $x - h$ equal to zero, and the y-coordinate of the vertex is k. The number a is the same as in the form $y = ax^2 + bx + c$ and has the following effects on the graph:

- If a is positive the parabola opens upward, and if a is negative the parabola opens downward.
- If the absolute value of a is less than one, the parabola is shrunk vertically to become "smashed flatter" than the parabola $y = x^2$.
- If the absolute value of a is greater than one, the parabola is stretched vertically, to become "narrower" than the parabola $y = x^2$.

◇ **Example 3.2(b):** Sketch the graph of $y = -4(x-2)^2 + 5$ on the same grid with a dashed graph of $y = x^2$. Label the vertex with its coordinates and make sure the shape of the graph compares correctly with that of $y = x^2$. Another Example

Solution: The value that makes the quantity $x - 2$ zero is $x = 2$, so that is the x-coordinate of the vertex, and the y-coordinate is $y = 5$. Because $a = -4$, the parabola opens downward and is narrower than $y = x^2$. From this we can graph the parabola as shown to the right.



Finding the Equation of a Parabola

If we are given the coordinates of the vertex of a parabola and the coordinates of one other point on the parabola, it is quite straightforward to find the equation of the parabola using the equation $y = a(x-h)^2 + k$. The procedure is a bit like what is done to find the equation of a line through two points, and we'll demonstrate it with an example.

◇ **Example 3.2(c):** Find the equation of the parabola with vertex $(-3, -4)$ with one x-intercept at $x = -7$.

Solution: Since the x-coordinate of the vertex is -3 , the factor $x-h$ must be $x+3$ so that $x = -3$ will make it zero. Also, k is the y-coordinate of the vertex, -4. From these two things we know the equation looks like $y = a(x+3)^2 - 4$. Now we need values of x and y for some other point on the parabola; if we substitute them in we can solve for a. Since -7 is an x-intercept, the y-coordinate that goes with it is zero, so we have the point $(-7, 0)$. Substituting it in we get

$$\begin{aligned} 0 &= a(-7+3)^2 - 4 \\ 4 &= (-4)^2 a \\ 4 &= 16a \\ \tfrac{1}{4} &= a \end{aligned}$$

The equation of the parabola is then $y = \frac{1}{4}(x+3)^2 - 4$.

Changing From Standard Form to Vertex Form

What we have seen is that if we have the equation of a parabola in vertex form $y = a(x-h)^2 + k$ it is easy to get a general idea of what the graph of the parabola looks like. Suppose instead that we have a quadratic function in standard form $y = ax^2 + bx + c$. If we could put it in the vertex form $y = a(x-h)^2 + k$, then we would be able to graph it easily. The key to doing this is the following:

> Suppose we have $x^2 + cx$. If we add $d = (\frac{1}{2}c)^2$ to this quantity the result $x^2 + cx + d$ factors into two *EQUAL* factors.

The process of adding such a $d = (\frac{1}{2}c)^2$ is called **completing the square**. Of course we can't just add anything we want to an expression without changing its value. The key is that if we want a new term, we can add it as long as we subtract it as well. Here's how we use this idea to put $y = x^2 + bc + c$ into vertex form:

$y = x^2 - 6x - 3$	
$y = x^2 - 6x + \_\_\_ - 3$	pull constant term away from the x^2 and x terms
$y = x^2 - 6x + 9 - 9 - 3$	multiply 6 by $\frac{1}{2}$ and square to get 9, add and subtract
$y = (x^2 - 6x + 9) + (-9 - 3)$	group terms to form a quadratic trinomial with "leftover constants" and "leftover constants"
$y = (x-3)^2 - 12$	factor the trinomial, combine constants

And here's how to do it when the a of $y = ax^2 + bx + c$ is not one:

◇ **Example 3.2(d):** Put the equation $y = -2x^2 + 12x + 5$ into the form $y = a(x-h)^2 + k$ by completing the square.

$y = -2x^2 + 12x + 5$	
$y = -2(x^2 - 6x) + 5$	factor -2 out of the x^2 and x terms
$y = -2(x^2 - 6x + 9 - 9) + 5$	add and subtract 9 inside the parentheses
$y = -2(x^2 - 6x + 9) + 18 + 5$	bring the -9 out of the parentheses - it comes out as $+18$ because of the -2 factor outside the parentheses
$y = -2(x-3)^2 + 23$	factor the trinomial, combine the added constants

---

Section 3.2 Exercises To Solutions

1. For each of the following, sketch the graph of $y = x^2$ and the given function on the same coordinate grid. The purpose of sketching $y = x^2$ is to show how "open" the graph of the given function is, *relative to* $y = x^2$.

   (a) $f(x) = 3(x-1)^2 - 7$

   (b) $g(x) = (x+4)^2 - 2$

   (c) $h(x) = -\frac{1}{3}(x+2)^2 + 5$

   (d) $y = -(x-5)^2 + 3$

2. For each of the following, find the equation of the parabola meeting the given conditions. Give your answer in the vertex form $y = a(x-h)^2 + k$.

   (a) Vertex $V(-5, -7)$ and through the point $(-3, -25)$.

   (b) Vertex $V(6, 5)$ and y-intercept 29.

   (c) Vertex $V(4, -3)$ and through the point $(-5, 6)$.

   (d) Vertex $V(-1, -4)$ and y-intercept 1.

3. For each of the following, use completing the square to write the equation of the function in the vertex form $f(x) = a(x-h)^2 + k$. Check your answers by graphing the original equation and your vertex form equation together on your calculator or other device - they should of course give the same graph if you did it correctly! Hint for (c): Divide (or multiply) both sides by -1.

   (a) $f(x) = x^2 - 6x + 14$

   (b) $g(x) = x^2 + 12x + 35$

   (c) $h(x) = -x^2 - 8x - 13$

   (d) $y = 3x^2 + 12x + 50$

4. Write the quadratic function $y = 3x^2 - 6x + 5$ in vertex form.

5. Write the quadratic function $h(x) = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{9}{4}$ in vertex form.

6. Sketch the graphs of $y = x^2$ and $f(x) = -\frac{3}{4}(x-1)^2 + 6$ without using your calculator, check with your calculator or other device.

7. Find the equation of the parabola with vertex $V(4, 1)$ and y-intercept -7. Give your answer in $y = a(x-h)^2 + k$ form.

8. Write the quadratic function $y = -3x^2 + 6x - 10$ in vertex form.

9. Find the equation of the quadratic function whose graph has vertex $V(3, 2)$ and passes through $(-1, 10)$. Give your answer in vertex form.

10. Write the quadratic function $y = 5x^2 + 20x + 17$ in vertex form.

3.3 Linear and Quadratic Inequalities

> **Outcome/Criteria:**
>
> 3. (f) Solve a linear inequality; solve a quadratic inequality.

Let's recall the notation $a > b$, which says that the number a is greater than the number b. Of course that is equivalent to $b < a$, b is less than a. So, for example, we could write $5 > 3$ or $3 < 5$, both mean the same thing. Note that it is not correct to say $5 > 5$, since five is not greater than itself! On the other hand, when we write $a \geq b$ it means a is greater than *or equal to* b, so we could write $5 \geq 5$ and it would be a true statement. Of course $a \geq b$ is equivalent to $b \leq a$.

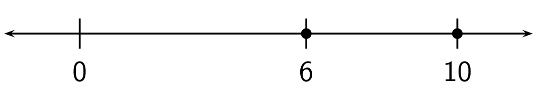
In this section we will consider algebraic inequalities; that is, we will be looking at inequalities that contain an unknown value. An example would be

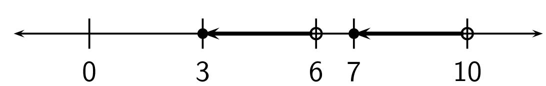
$$3x + 5 \leq 17,$$

and our goal is to *solve* the inequality. This means to find all values of x that make this true. This particular inequality is a **linear inequality**; before solving it we'll take a quick look at how linear inequalities behave.

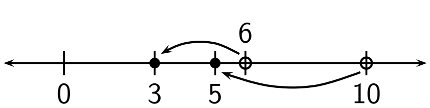
Solving Linear Inequalities

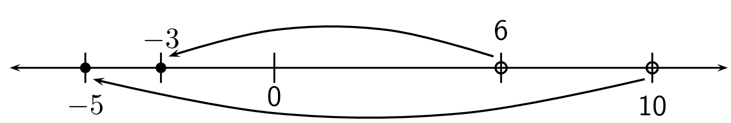
Consider the *true* inequality $6 < 10$; if we put dots at each of these values on the number line the larger number, ten, is farther to the right, as shown to the left below. If we subtract, say, three from both sides of the inequality we get $3 < 7$. The effect of this is to shift each number to the left by five units, as seen in the diagram below and to the right, so their relative positions don't change.





Suppose now that we divide both sides of the same inequality $6 < 10$ by two. We then get $3 < 5$, which is clearly true, and shown on the diagram below and to the left. If on the other hand, we divide both sides by -2 we get $-3 < -5$. This is not true! The diagram below and to the right shows what is going on here; when we divide both sides by a negative we reverse the positions of the two values on the number line, so to speak.





The same can be seen to be true if we multiply both sides by a negative. This leads us to the following principle.

Solving Linear Inequalities

To solve a linear inequality we use the same procedure as for solving a linear equation, *except that we reverse the direction of the inequality whenever we multiply or divide BY a negative.*

◇ **Example 3.3(a):** Solve the inequality $3x + 5 \leq 17$.

Solution: We'll solve this just like we would solve $3x + 5 = 17$. At some point we will divide by 3, but since we are not dividing by a negative, we won't reverse the inequality.

$$\begin{aligned} 3x + 5 &\leq 17 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}$$

This says that the original inequality is true for any value of x that is less than or equal to 4. Note that zero is less than 4 and makes the original inequality true; this is a check to make sure at least that the inequality in our answer goes in the right direction.

◇ **Example 3.3(b):** Solve $-3x + 7 > 19$.

Solution: Here we can see that at some point we will divide by -3, and *at that point* we'll reverse the inequality.

$$\begin{aligned} -3x + 7 &> 19 \\ -3x &> 12 \\ x &< -4 \end{aligned}$$

This says that the original inequality is true for any value of x that is less than -4. Note that -5 is less than -4 and makes the original inequality true, so the inequality in our answer goes in the right direction.

Solving Quadratic Inequalities

Suppose that we wish to know where the function $f(x) = -x^2 + x + 6$ is positive. Remember that what this means is we are looking for the x values for which the corresponding values of $f(x)$ are greater than zero; that is, we want $f(x) > 0$. But $f(x)$ is $-x^2 + x + 6$, so we want to solve the inequality

$$-x^2 + x + 6 > 0.$$

Let's do this by first multiplying both sides by -1 (remembering that this causes the direction of the inequality to change!) and then factoring the left side:

$$\begin{aligned} -x^2 + x + 6 &> 0 \\ x^2 - x - 6 &< 0 \\ (x - 3)(x + 2) &< 0 \end{aligned}$$

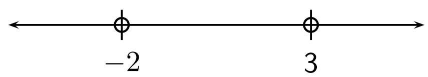
Now if $(x-3)(x+2)=0$, it must be the case that either $x-3=0$ or $x+2=0$. Unfortunately, solving the inequality is not so simple¡ but the method we will use in not that hard, either:

- Get zero on one side of the inequality, and a quadratic expression on the other side. In some way, get the coefficient a of x^2 to be positive.
- Factor the quadratic expression to find the values that make it zero.
- Plot the values that make the factors zero on a number line, as open circles if the inequality is *strict* ($<$ or $>$), or as solid dots if not ($\leq$ or $\geq$).
- The two (or occasionally one) points divide the number line into three (two) intervals. Test a value in each interval to see if the inequality holds there; if it does, shade that interval.
- Describe the shaded interval(s) using interval notation.

Let's illustrate by finishing solving the above inequality and with another example.

◇ **Example 3.3(c):** Solve the inequality $-x^2+x+6>0$. Another Example

Solution: We begin by multiplying both sides by $-$ to get the coefficient of x^2 to be positive, and we then factor the quadratic expression. These steps are shown above. The values that make the quadratic expression zero are 3 and -2, so we plot those on a number line. Because the inequality is not true for those particular values of x, we plot them as open circles:



Note that the two points -2 and 3 divide the number line into the three intervals $(-\infty,-2)$, $(-2,3)$ and $(3,\infty)$. For each interval, the inequality will be either true or false throughout the entire interval. We will test one value in each interval - in the first interval lets test $x=-3$, in the second interval $x=0$ and in the third interval $x=4$. (Note that the inequality we are testing is $(x-3)(x+2)<0$, which is equivalent to the original inequality.) All we need to test for is whether the inequality is true, so we only care about are whether things are positive or negative:

- For $x=-3$, $(x-3)(x+2)=(-3-3)(-3+2)=(-)(-)=+$, so the inequality $(x-3)(x+2)<0$ is false in the interval $(-\infty,-2)$.
- For $x=0$, $(x-3)(x+2)=(-3)(+2)=(-)(+)=-$, so $(x-3)(x+2)<0$ is true in the interval $(-2,3)$.
- For $x=4$, $(x-3)(x+2)=(4-3)(4+2)=(+)(+)=-$, so $(x-3)(x+2)<0$ is false in the interval $(3,\infty)$.

Based on the above, we can shade the portion of our number line where the inequality is true:

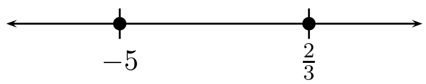
The solution to the inequality is then the interval $(-2,3)$.

◇ **Example 3.3(d):** Solve the inequality $3x^2 + 13x \geq 10$.

Solution: First we get zero on one side and factor the quadratic inequality:

$$\begin{aligned} 3x^2 + 13x &\geq 10 \\ 3x^2 + 13x - 10 &\geq 0 \\ (3x-2)(x+5) &\geq 0 \end{aligned}$$

The values that make $(3x-2)(x+5)$ zero are $x = -5$ and $x = \frac{2}{3}$. (The second of these is found by setting $3x - 2$ equal to zero and solving.) We plot those values on a number line, as *solid* dots because the inequality is greater than *or equal to*:



We then test, for example, $x = -6$, $x = 0$ and $x = 1$. When $x = -6$,

$$(3x-2)(x+5) = (-18-2)(-6+5) = (-)(-) = +,$$

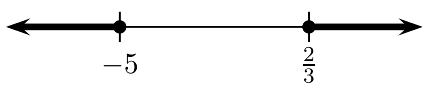
so the inequality $(3x-2)(x+5) \geq 0$ *IS* true for that value. For $x = 0$ we have

$$(3x-2)(x+5) = (-2)(+5) = (-)(+) = -,$$

so the inequality is not true for that value and, for $x = 1$,

$$(3x-2)(x+5) = (3-2)(1+5) = (+)(+) = +$$

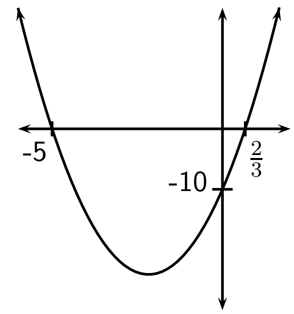
so the inequality is true for $x = 1$. Based on all this we can shade our number line:



This shows that the solution to the inequality is $(-\infty, -5] \cup [\frac{2}{3}, \infty)$.

NOTE: You may have guessed from the above that when we are solving a quadratic inequality for a number line divided into three parts, we can actually test a value in just one of the three intervals. The sign of the quadratic expression then changes as we move from one interval to any *adjacent* interval. So for the example we just did, we could have tested $x = 1$ and found that $(3x-2)(x+5)$ is positive in the interval $(\frac{2}{3}, \infty)$. The expression is then negative in the next interval to the left, $(-5, \frac{2}{3})$ and positive again in $(-\infty, -5)$. (I did not use [and] here because we are talking about where the expression is positive, not where the inequality is true.) We will see inequalities later where this approach does not necessarily work, so it is perhaps best to test every interval. Doing so can also alert you to any mistakes you might make in determining the sign within an interval.

To the right is the graph of $f(x) = 3x^2 + 13x - 10$. Compare the solution set from the last example with the graph of the function to see how the solution relates to the graph. The shaded number line shows us the intervals where the function is positive or negative - remember when looking at the graph that "the function" means the y values.



We can also use our knowledge of what the graph of a quadratic function looks like to solve an inequality, as demonstrated in the following example.

◇ **Example 3.3(e):** Solve the quadratic inequality $5 > x^2 + 4x$.

Solution: The given inequality is equivalent to $0 > x^2 + 4x - 5$. Setting $y = x^2 + 4x - 5$ and factoring gives us $y = (x + 5)(x - 1)$. Therefore the graph of y is a parabola, opening upward and with x-intercepts -5 and 1. y is then less than zero between -5 and 1, not including either. The solution to the inequality is then $-5 < x < 1$ or $(-5, 1)$.

Section 3.3 Exercises To Solutions

1. Solve each of the following linear inequalities.

   (a) $5x + 4 \geq 18$ (b) $7 - 2x \leq 13$

   (c) $4(x + 3) \geq x - 3(x - 2)$ (d) $4x + 2 > x - 7$

   (e) $5x - 2(x - 4) > 35$ (f) $5y - 2 \leq 9y + 2$

2. Solve each of the following inequalities. (**Hint:** Get zero on one side first, if it isn't already.) Give your answers as inequalities, OR using interval notation.

   (a) $x^2 + 8 < 9x$ (b) $2x^2 \leq x + 10$

   (c) $\frac{2}{3}x^2 + \frac{7}{3}x \leq 5$ (d) $5x^2 + x > 0$

   (e) $2x + x^2 < 15$ (f) $x^2 + 7x + 6 \geq 0$

3. Try solving each of the following inequalities using a method similar to that shown in this section.

   (a) $(x - 3)(x + 1)(x + 4) < 0$ (b) $(x - 5)(x + 2)(x - 2) \geq 0$

   (c) $(x - 3)^2(x + 2) > 0$ (d) $(x - 2)^3 \leq 0$

4. For each of the following, factor to solve the inequality.

   (a) $x^3 + 3x^2 + 2x \geq 0$

   (b) $10x^3 < 29x^2 + 21x$

5. (a) Determine **algebraically** where $x^2 - 4x + 4 > 0$. Then check your answer by graphing $f(x) = x^2 - 4x + 4$ and seeing where the function is positive.

   (b) Determine where $-x^2 - 6x - 11 \geq 0$. Check this answer by graphing the function $g(x) = -x^2 - 6x - 11$.

6. Consider the equation $\frac{1}{2}(x+1)^2 = 2$. One way to solve it would be to expand the left side (by "FOILing" it out and multiplying by $\frac{1}{2}$), then getting zero on one side and factoring. In this exercise you will solve it in a more efficient manner.

   - Begin by eliminating the fraction on the left, in the same way that we always do this.
   - Take the square root of both sides, remembering that you need to put a $\pm$ on the right.
   - You should be able to finish it from here! Check your answers (there are two) in the original equation.

7. (a) Determine the interval or intervals on which $\frac{1}{3}(x-2)^2 < 3$.

   (b) Graph $y = \frac{1}{3}(x-2)^2$ and $y = 3$ together on the same grid, using technology. Your answer to (a) should be the x values for which the graph of $y = \frac{1}{3}(x-2)^2$ is below the graph of $y = 3$.

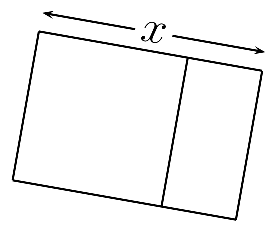
3.4 Applications of Quadratic Functions

Outcome/Criteria:

3. (g) Solve a problem using a given quadratic model; create a quadratic model for a given situation.

Quadratic functions arise naturally in a number of applications. As usual, we often wish to know the output for a given input, or what inputs result in a desired output. However, there are other things we may want to know as well, like the maximum or minimum value of a function, or when the function takes values larger or smaller than some given value. Let's begin by revisiting the following example:

◇ **Example 2.5(b):** A farmer is going to create a rectangular field with two "compartments", as shown to the right. He has 2400 feet of fence with which to do this. Letting x represent the dimension shown on the diagram, write an equation for the total area A of the field as a function of x, simplifying your equation as much as possible. Give the feasible domain for the function.



We determined that the equation for the area as a function of x is $A = 800x - \frac{2}{3}x^2$ for $0 < x < 1200$.

◇ **Example 3.4(a):** Determine the maximum area of the field, and the value of x for which it occurs.

Solution: Rearranging the function to get $A = -\frac{2}{3}x^2 + 800x$, we can see that the graph would be a parabola opening downward, so the maximum will be the A-coordinate of the vertex. The x-coordinate of the vertex is given by

$$x = -\frac{800}{2(-\frac{2}{3})} = \frac{800}{\frac{4}{3}} = (800)(\tfrac{3}{4}) = 600 \text{ feet}$$

Substituting this into the function we get

$$A = -\tfrac{2}{3}(600)^2 + 800(600) = 240,000 \text{ square feet}$$

The maximum area is $240,000$ square feet, which is attained when $x = 600$ feet.

◇ **Example 3.4(b):** Determine the values of x for which the area is $100,000$ square feet or more.

Solution: Here we need to solve the inequality

$$-\frac{2}{3}x^2 + 800x \geq 100,000.$$

We first change the inequality to equal and multiply both sides by three to clear the fraction. This gives us $-2x^2 + 2400x = 300,000$. We can then move the $300,000$ to the left side and divide by -2 to get $x^2 - 1200x + 150,000 = 0$. It is possible that this factors, but let's use the quadratic formula to solve for x:

$$x = \frac{-(-1200) \pm \sqrt{(-1200)^2 - 4(1)(150,000)}}{2(1)} = \frac{1200 \pm \sqrt{840,000}}{2} = 141.7, 1058.3$$

As stated in the previous example, the graph of the function is a parabola opening down, so the function has value greater than $100,000$ between the two values that we found. Thus the area of the field is greater than or equal to $100,000$ square feet when x is between 141.7 and 1058.3 feet.

Section 3.4 Exercises

To Solutions

1. The height h (in feet) of an object that is thrown vertically upward with a starting velocity of 48 feet per second is a function of time t (in seconds) after the moment that it was thrown. The function is given by the equation $h(t) = 48t - 16t^2$.

   (a) How high will the rock be after 1.2 seconds? Round your answer to the nearest tenth of a foot.

   (b) When will the rock be at a height of 32 feet? You should be able to do this by factoring, and you will get two answers; explain why this is, physically.

   (c) Find when the rock will be at a height of 24 feet. Give your answer rounded to the nearest tenth of a second.

   (d) When does the rock hit the ground?

   (e) What is the mathematical domain of the function? What is the feasible domain of the function?

   (f) What is the maximum height that the rock reaches, and when does it reach that height?

   (g) For what time periods is the height less than 15 feet? Round to the nearest hundredth of a second.

2. In statistics, there is a situation where the expression $x(1-x)$ is of interest, for $0 \leq x \leq 1$. What is the maximum value of the function $f(x) = x(1-x)$ on the interval $[0, 1]$? Explain how you know that the value you obtained is in fact a maximum value, not a minimum.

3. Use the quadratic formula to solve $s = s_0 + v_0 t - \frac{1}{2}gt^2$ for t. (**Hint:** Move everything to the left side and apply the quadratic formula. Keep in mind that t is the variable, and all other letters are constants.)

4. The Acme Company also makes and sells Geegaws, in addition to Widgets and Gizmos. Their weekly profit P (in dollars) is given by the equation $P = -800 + 27x - 0.1x^2$, where x is the number of Geegaws sold that week. Note that it will be possible for P to be negative; of course negative profit is really loss!

   (a) What is the P-intercept, *with units*, of the function? What does it mean, "in reality"?

   (b) Find the numbers of Geegaws that Acme can make and sell in order to make a true profit; that is, we want P to be positive. Round to the nearest whole Geegaw.

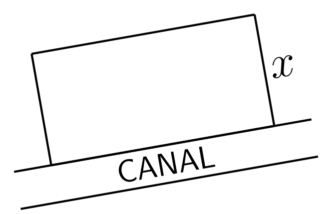
   (c) Find the maximum profit they can get. How do we know that it is a maximum, and not a minimum?

5. In a previous exercise you found the revenue equation $R = 20000p - 100p^2$ for sales of Widgets, where p is the price of a widget and R is the revenue obtained at that price.

   (a) Find the price that gives the maximum revenue, and determine the revenue that will be obtained at that price.

   (b) For what prices will the revenue be $500,000 or less?

6. Another farmer is going to create a rectangular field (without compartments) against a straight canal, by putting fence along the three straight sides of the field that are away from the canal. (See picture to the right.) He has 1000 feet of fence with which to do this.



   (a) Write an equation (in simplified form) for the area A of the field, in terms of x.

   (b) Give the feasible domain of the function.

   (c) Determine the maximum area of the field.

   (d) Determine all possible values of x for which the area is at least 90,000 square feet.

3.5 Distance, Midpoint and Circles

Performance Criteria:

3. (h) Find the distance between two points in the plane; find the midpoint of a segment between two points in a plane.

   (i) Graph a circle, given its equation in standard form. Given the center and radius of a circle, give its equation in standard form.

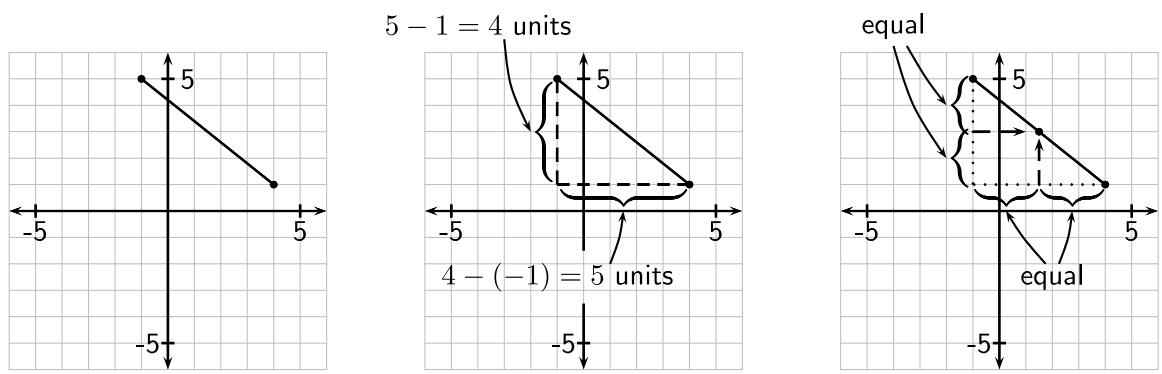
   (j) Put the equation of a circle in standard form

$$(x-h)^2 + (y-k)^2 = r^2$$

   by completing the square.

Length and Midpoints of a Segment

In this section we will consider some geometric ideas that are not central to the main ideas of this course, but which are occasionally of interest. Consider a **segment** in the xy-plane like the one from $(-1, 5)$ to $(4, 1)$, shown to the below and to the left. (A segment is a section of a line that has ends in both directions.) There are two things we'd like to know about this segment: its length, and the coordinates of its midpoint.



To find the length of the segment we can think of a right triangle whose legs are shown by the dashed segments in the middle picture above, and apply the Pythagorean theorem $a^2 + b^2 = c^2$. In this case we'll let a be the length of the horizontal leg; we can see that its length is five units, which can be obtained by subtracting the x-coordinates: $4 - (-1) = 5$. b is then the length of the vertical leg, given by $5 - 1 = 4$. Substituting into the quadratic formula gives $5^2 + 4^2 = c^2$, so $c = \sqrt{5^2 + 4^2} = \sqrt{41}$. (We don't need the usual $\pm$ with the root because c must be positive in this situation, because it is a length.)

To find the midpoint we use an idea that should be intuitively clear: The x-coordinate of the midpoint should be halfway between the x-coordinates of the endpoints, and the y-coordinate of the midpoint should be halfway between the y-coordinates of the endpoints. This is shown in the picture above and to the right. To find a value halfway between two other values *we average the two values*. The x-coordinate of the midpoint is then the average of -1 and 4, which is

$x = \dfrac{-1+4}{2} = 1\frac{1}{2}$, and the y-coordinate of the midpoint is $y = \dfrac{1+5}{2} = 3$. The coordinates of the midpoint are then $\left(1\frac{1}{2}, 3\right)$.

We can summarize the general procedure for finding the length and midpoint of a segment as follows:

Length and Midpoint of a Segment

- To find the length of a segment between two points P and Q, subtract the x-coordinates to find the horizontal distance (or negative of the distance, it doesn't matter) between the points. Then subtract the y-coordinates to find the vertical distance. Substitute those two distances as a and b of the Pythagorean formula $a^2 + b^2 = c^2$; the *positive* value of c is the length of the segment.
- To find the coordinates of the midpoint of a segment, average the x-coordinates of the endpoints to get the x-coordinate of the midpoint. Average the y-coordinates of the endpoints to get the y-coordinate of the midpoint.
- The above two things can be done symbolically as follows. For a segment with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$, the length l and midpoint M of the segment can be found by

$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad \text{and} \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

◇ **Example 3.5(a):** Find the length and midpoint of the segment with endpoints $(-5, -3)$ and $(7, 10)$. Give the length in exact form *and* as a decimal rounded to the nearest tenth of a unit. Midpoint Example

Solution: The length is

$$l = \sqrt{(-5-7)^2 + (-3-10)^2} = \sqrt{(-12)^2 + (-13)^2} = \sqrt{144 + 169} = \sqrt{313} = 17.7$$

and the midpoint is

$$M = \left(\frac{-5+7}{2}, \frac{-3+10}{2}\right) = \left(\frac{2}{2}, \frac{7}{2}\right) = (1, 3\tfrac{1}{2})$$

◇ **Example 3.5(b):** One endpoint of a segment is $(5,-4)$ and the midpoint is $(7,3)$. Find the coordinates of the other endpoint.

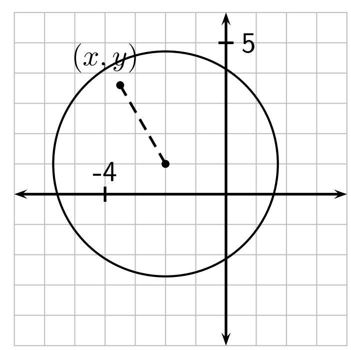
Solution: Note that we know one of (x_1,y_1) or (x_2,y_2) but not both, and we also know the midpoint. If we take $(5,-4)$ to be (x_2,y_2) we have that

$$(7,3)=\left(\frac{x_1+5}{2},\frac{y_1+(-4)}{2}\right)$$

This gives us the two equations $7=\dfrac{x_1+5}{2}$ and $3=\dfrac{y_1+(-4)}{2}$. Solving these for x_1 and y_1 gives us $x_1=9$ and $y_1=10$, so the missing endpoint is then $(9,10)$.

Circles in the Plane

Consider the circle shown to the right. It is easily seen that the center of the circle is the point $(-2,1)$ and the radius is three units. The point (x,y) is an arbitrary point on the circle; regardless of where it is on the circle, its distance from the center is three units. We can apply the Pythagorean theorem or the formula for the length of a segment to the dashed segment to get



$$[x-(-2)]^2+(y-1)^2=3^2 \tag{1}$$

This equation must be true for any point (x,y) on the circle, and any ordered pair (x,y) that makes the equation true *MUST* be on the circle. Therefore we say that (1) is the equation of the circle. In general the equation of the circle with center (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$. We usually do a little simplification; we would simplify equation (1) to $(x+2)^2+(y-1)^2=9$.

Equation of a Circle

The equation of the circle with center (h,k) and radius r is

$$(x-h)^2+(y-k)^2=r^2$$

◇ **Example 3.5(c):** Give the equation of the circle with center $(4,-7)$ and radius 6.

Solution: By the above, the equation would be $(x-4)^2+(y-(-7))^2=6^2$, which can be cleaned up a little to get $(x-4)^2+(y+7)^2=36$. We don't usually multiply out the two terms $(x-4)^2$ and $(y+7)^2$.

This last example illustrates where formality is sometimes a bit confusing. Where we see a minus sign $(y-k)$ in the equation $(x-h)^2+(y-k)^2=r^2$, in this last example we have a plus sign,

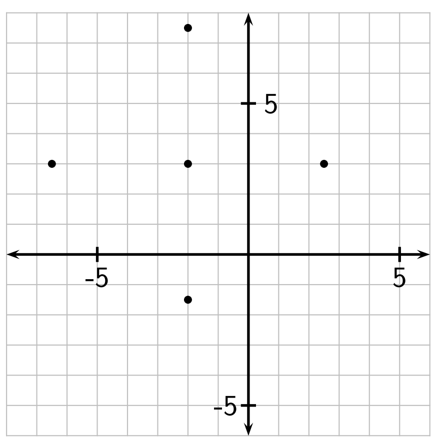
$(y+7)$. Here is a nice way to think about this concept: Recall that when solving the equation $x^2 - 2x - 15 = 0$ we factor to get $(x-5)(x+3) = 0$. The solutions are then the values that make the two factors $(x-5)$ and $(x+3)$ equal to zero, 5 and -3. Note also that for the equation $(x-4)^2 + (y+7)^2 = 36$ from Example 3.5(c), the values of x and y that make $(x-4)$ and $(y+7)$ equal to zero are the coordinates of the center of the circle. These two things again illustrate the important general principle we first saw in Section 1.7:

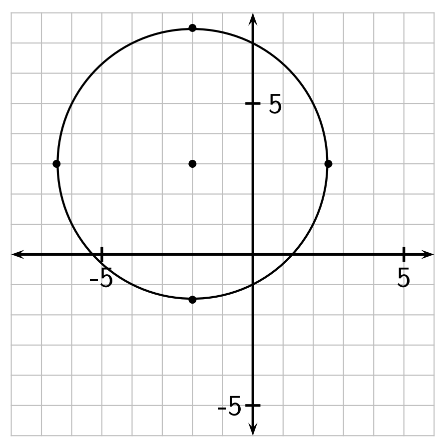
> *When we have a quantity $(x \pm a)$, the value of x (or any other variable in that position) that makes the entire quantity zero is likely an important value of that variable.*

Suppose we know that the equation $(x-4)^2 + (y+7)^2 = 36$ is the equation of a circle, and we recognize that the values $x = 4$ and $y = -7$ that make $(x-4)$ and $(y+7)$ both zero are probably important. For a circle, there is only one point that is more important than any others, the center. Therefore $(4, -7)$ must be the center of the circle.

◇ **Example 3.5(d):** Sketch the graph of the circle with equation $(x+2)^2 + (y-3)^2 = 20$. Correctly locate the center and the four points that are horizontally and vertically aligned with the center. (The locations of the four points will have to be somewhat approximate.)

Solution: The center of the circle is $(-2, 3)$ since $x = -2$ and $y = 3$ make $(x+2)$ and $(y-3)$ equal to zero. Also, $r^2 = 20$ so $r = \sqrt{20} \approx 4.5$. (We ignore the negative square root because the radius is a positive number.) To graph the circle we first plot the center, then plot four points that are 4.5 units to the right and left of and above and below the center. Those points are shown on the graph below and to the left. We then sketch in the circle through those four points and centered at $(-2, 3)$. This is shown below and to the right.





Putting the Equation of a Circle in Standard Form

Consider the equation $x^2 + y^2 + 2x - 10y + 22 = 0$. It turns out this is the equation of a circle, but it is not in the form that we have been working with, so we can't easily determine the center and radius. What we would like to do is put the equation into the standard form $(x-h)^2+(y-k)^2=r^2$. To do this we use the **completing the square** procedure that we saw in Section 3.2. Remember the key idea:

> Suppose we have x^2+cx. If we add $d=(\frac{1}{2}c)^2$ to this quantity the result x^2+cx+d factors into two *EQUAL* factors.

When using this idea with circles there will one slight change; Rather than adding and subtracting the same value on one side of an equation (which is, of course, equivalent to adding zero), we'll add the desired value $d=(\frac{1}{2}c)^2$ to both sides of the equation. That will keep the equation "in balance."

◇ **Example 3.5(e):** $x^2+y^2+2x-10y+22=0$ is the equation of a circle. Put it into standard form, then give the center and radius of the circle. Another Example

Solution:

$x^2+y^2+2x-10y+22$	$=$ 0	the original equation
$x^2+y^2+2x-10y$	$=$ -22	subtract 22 from both sides
x^2+2x+y^2-10y	$=$ -22	rearrange terms on the left side to get the x terms together and the y terms together
$x^2+2x+1+y^2-10y+25$	$=$ $-22+1+25$	add $[\frac{1}{2}(2)]^2=1$ and $[\frac{1}{2}(10)]^2=25$ to both sides
$(x+1)(x+1)+(y-5)(y-5)$	$=$ 4	factor the left side and simplify the right
$(x+1)^2+(y-5)^2$	$=$ 4	equation in standard form

The circle has center $(-1,5)$ and radius 2.

Section 3.5 Exercises

To Solutions

1. (a) Find the distance from $(-3,1)$ to $(5,7)$.

   (b) Find the midpoint of the segment from $(-3,1)$ to $(5,7)$.

2. (a) Find the distance from $(-1,8)$ to $(0,1)$. Give your answer in simplified square root form *AND* as a decimal, rounded to the nearest hundredth.

   (b) Find the midpoint of the segment from $(-1,8)$ to $(0,1)$.

3. (a) Find the distance from $(1.3, -3.6)$ to $(5.9, 2.1)$. Give your answer as a decimal, rounded to the nearest hundredth.

   (b) Find the midpoint of the segment from $(1.3, -3.6)$ to $(5.9, 2.1)$.

4. Find the other endpoint of the segment with endpoint $(-3, -7)$ and midpoint $(-1, 2\frac{1}{2})$.

5. Find the other endpoint of the segment with endpoint $(5, -2)$ and midpoint $(0, 0)$. Try to do this with no calculation; visualize a graph of the segment.

6. Determine the equation of the circle with the given center and radius.

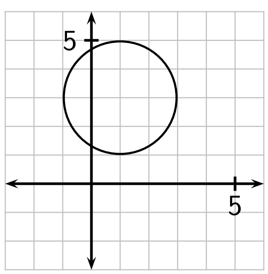
   (a) $(2, 3)$, $r = 5$ (b) $(2, -3)$, $r = 5$ (c) $(0, -5)$, $r = 6$.

7. Give the center and radius for each of the circles whose equations are given. When the radius is not a whole number, give the radius in both exact form and as a decimal, rounded to the the nearest tenth.

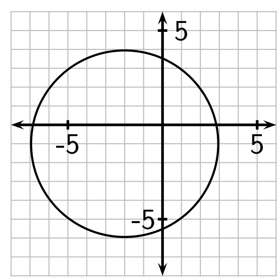
   (a) $(x + 3)^2 + (y - 5)^2 = 4$ (b) $(x - 4)^2 + y^2 = 28$ (c) $x^2 + y^2 = 7$

8. Give the equation of each circle shown below.

   (a)



   (b)



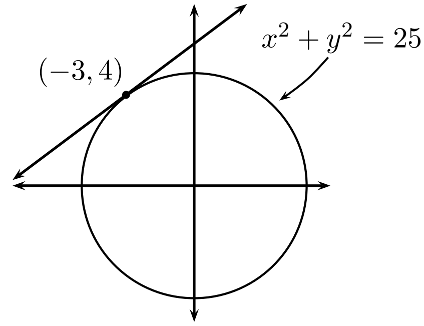
9. Put each equation into standard form by completing the square, then give the center and radius of each circle.

   (a) $x^2 + y^2 = -2x + 6y + 6$ (b) $x^2 - 4x + y^2 + 6y = 12$

   (c) $x^2 + y^2 - 6x + 2y = 18$ (d) $x^2 + y^2 + 10x + 15 = 0$

10. Determine an equation for the circle that has a diameter with end points $A(7, 2)$ and $B(-1, 6)$. (The diameter is a segment whose endpoints are on the circle and that passes through the center of the circle.)

11. A line that touches a circle in only one point is said to be **tangent** to the circle. The picture to the right shows the graph of the circle $x^2 + y^2 = 25$ and the line that is tangent to the circle at $(-3, 4)$. Find the equation of the tangent line. (**Hint:** The tangent line is perpendicular to the segment that goes from the center of the circle to the point $(-3, 4)$.)



12. Equations of circles are not functions, so we don't generally talk about a maximum or minimum in the case of a circle. Determine the highest and lowest points (that is, the points with largest and smallest y values) on the circle with equation $(x - 2)^2 + (y + 3)^2 = 49$.

13. A line is called a **perpendicular bisector** of a segment if the line intersects the segment at its midpoint and the line is perpendicular to the segment. Determine the equation of the perpendicular bisector of the segment with endpoints $(1, -1)$ and $(5, 5)$.

14. Determine the equation of the circle for which the segment with endpoints $(1, -1)$ and $(5, 5)$ is a diameter. (A **diameter** is a segment passing through the center of the circle, and whose endpoints are on the circle.)

3.6 Systems of Two Non-Linear Equations

Performance Criteria:

3. (k) Solve a system of two non-linear equations.

Let's begin by considering the system of equations

$$\begin{aligned} x^2+y &= 9 \\ x-y+3 &= 0 \end{aligned}$$

The second equation is linear, but the first is not, its graph is a parabola. If you think of a line and a parabola floating around in the xy-plane, they might not cross at all, they might touch in just one point, or they might cross in two places. (Try drawing pictures of each of these situations.) In this section we will see how to solve systems of equations like this one, where one or both of the equations are non-linear. In some cases the equations may not even represent functions.

Of course solving a system of equations means that we are looking for (x, y) pairs that make *BOTH* equations true. There are multiple options for going about this in this case:

- Solve the first equation for y and substitute into the second equation.
- Solve the second equation for x or y and substitute into the first equation.
- Add the two equations.
- Solve both equations for y and set the results equal to each other.

The least desirable of these options would be to solve the second equation for x, because we would get $x = y-3$, and substituting that into the first equation would give $(y-3)^2+y = 9$. This would require squaring $y-3$, which is not hard, but the other choices above would not require such a computation. Let's take a look at how a couple of those options would go.

◇ **Example 3.6(a):** Solve $\begin{aligned} x^2+y &= 9 \\ x-y+3 &= 0 \end{aligned}$ by solving the second equation for y and substituting into the first equation.

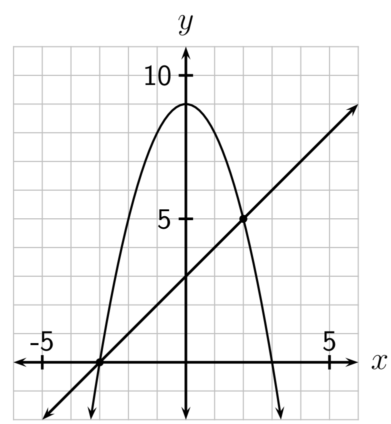
Solution: Solving the second equation for y gives $y = x+3$. Substituting that into the first equation we get $x^2+x+3 = 9$. Subtracting nine from both sides and factoring leads to the solutions $x = -3$ and $x = 2$. We can substitute these values into either equation to find the corresponding values of y. Using the second equation, when $x = -3$ we get $y = 0$, and when $x = 2$, $y = 5$. The solutions to the system are then $(-3, 0)$ and $(2, 5)$.

NOTE: When stating your answer *you must make it clear which values of x and y go together*. The easiest way to do this is probably just giving the solutions as ordered pairs, as done in the above example.

◇ **Example 3.6(b):** Solve $\begin{aligned} x^2+y &= 9 \\ x-y+3 &= 0 \end{aligned}$ by the addition method.

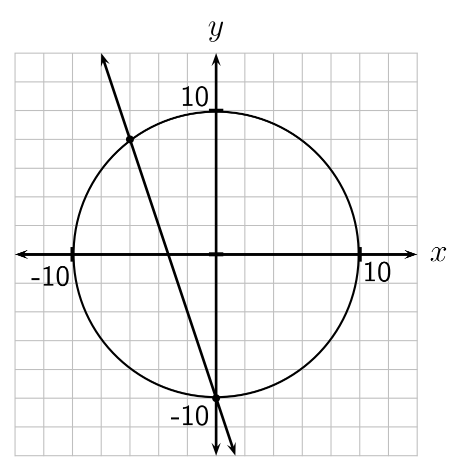
Solution: Adding the two equations gives $x^2+x+3=9$, the same equation we arrived at in the previous example. The remainder of the steps are exactly the same as in that example.

Recall that when a system of two *linear* equations in two unknowns has a solution, it is the (x,y) pair representing the point where the graphs of the two lines cross. In this case the solution is the point where the graphs of $x^2+y=9$ and $x-y+3=0$ cross. Note that these can both be solved for y to get $y=-x^2+9$ and $y=x+3$. Clearly the graph of the second is a line with slope one and y-intercept three. From what we learned in Section 2.6, the graph of $y=-x^2+9$ is the graph of $y=x^2$ reflected across the x-axis and shifted up nine units. Both of the equations are graphed together on the grid to the right, and we can see that the graphs cross at $(-3,0)$ and $(2,5)$, the solutions to the system.



◇ **Example 3.6(c):** Consider the two equations $3x+y=-10$ and $x^2+y^2=100$. Determine the solution(s) to the system of equations first by graphing both on the same grid, then by solving algebraically.

Solution: Solving the first equation for y gives $y=-3x-10$, a line with slope of -3 and y-intercept -10. The second equation is that of a circle with center at the origin and radius ten. Both of these are graphed together on the grid to the right, where we can see that the solutions to the system are $(0,-10)$ and $(-6,8)$. To solve the system algebraically we would solve the first equation for y, which we've already done, and substitute the result into the second equation:



$$\begin{aligned} x^2+(-3x-10)^2 &= 100 \\ x^2+9x^2+60x+100 &= 100 \\ 10x^2+60x &= 0 \\ 10x(x+6) &= 0 \end{aligned}$$

Therefore $x=0$ or $x=-6$. Substituting $x=0$ into $y=-3x-10$ results in $y=-10$, so $(0,-10)$ is a solution to the system. Substituting $y=-6$ into the same equation results in $x=8$, giving the solution $(-6,8)$.

Section 3.6 Exercises

To Solutions

1. Solve the system $\begin{aligned} x^2 + y &= 9 \\ x - y + 3 &= 0 \end{aligned}$ a third way, by solving the second equation for x and substitution the result into the first equation. (Earlier we said this isn't quite as efficient as the methods shown in Examples 3.6(a) and (b), but it should give the same results.)

2. Consider now the non-linear system of equations $\begin{aligned} x^2 + y &= 9 \\ 2x + y &= 10 \end{aligned}$.

   (a) Solve this system. How does your solution differ from that of Exercise 1 and the examples?

   (b) What would you expect to see if you graph the two equations for this system together? Graph them together to see if you are correct.

3. Do the following for the system $\begin{aligned} x + 3y &= 5 \\ x^2 + y^2 &= 25 \end{aligned}$.

   (a) Graph the two equations on the same grid. (Try doing this without using your calculator.) What does it appear that the solution(s) to the system is (are)?

   (b) Solve the system algebraically. If the results do not agree with your graph, try to figure out what is wrong.

4. For the system $\begin{aligned} x^2 + y^2 &= 8 \\ y - x &= 4 \end{aligned}$,

   (a) determine the solution(s) to the system by graphing the two equations on one grid, then

   (b) solve the system algebraically. Make sure your two answers agree, or find your error!

5. There is a particular way of using the substitution method for solving a system of equations that often comes up. It arises when we are solving a system that is given with both equations already solved for y, like the system $\begin{aligned} y &= x^2 - 4x + 1 \\ y &= 2x - 7 \end{aligned}$. What we say is that we "set the two expressions for y equal," which is of course the same as substituting the value of y from one equation into y in the other equation. Do this to solve this system.

6. Consider the system $\begin{aligned} x + y^2 &= 3 \\ x^2 + y^2 &= 5 \end{aligned}$.

   (a) Solve the system algebraically - you should get *four* solution pairs.

   (b) Graph the two equations on the same grid "by hand." Then check your graph and your answer to (a) by graphing the two equations with a device.

7. The Acme company manufactures Doo-dads. If x is the number of Doo-dads they make and sell in a week, their costs for the week (in dollars) are given by $C = 2450 + 7x$, and their revenues (also in dollars) are $R = 83x - 0.1x^2$. Find the numbers of Doo-dads that are made and sold when they break even (revenues equal costs).

3.7 Average Rates of Change, Secant Lines and Difference Quotients

Performance Criteria:

3. (l) Find the average rate of change of a function over an interval, from either the equation of the function or the graph of the function. Include units when appropriate.

   (m) Find and simplify a difference quotient.

Average Rate of Change

◇ **Example 3.7(a):** Suppose that you left Klamath Falls at 10:30 AM, driving north on on Highway 97. At 1:00 PM you got to Bend, 137 miles from Klamath Falls. How fast were you going on this trip?

Solution: Since the trip took two and a half hours, we need to divide the distance 137 miles by 2.5 hours, with the units included in the operation:

$$\text{speed} = \frac{137 \text{ miles}}{2.5 \text{ hours}} = 54.8 \text{ miles per hour}$$

Of course we all know that you weren't going 54.8 miles per hour the entire way from Klamath Falls to Bend; this speed is the *average* speed during your trip. When you were passing through all the small towns like Chemult and La Pine you likely slowed down to 30 or 35 mph, and between towns you may have exceeded the legal speed limit. 54.8 miles per hour is the speed you would have to go to make the trip in two and a half hours driving at a constant speed for the entire distance. The speed that you see any moment that you look at the speedometer of your car is called the *instantaneous* speed.

Speed is a quantity that we call a **rate of change**. It tells us the change in distance (in the above case, measured in miles) for a given change in time (measured above in hours). That is, for every change in time of one hour, an additional 54.8 miles of distance will be gained. In general, we consider rates of change when one quantity depends on another; we call the first quantity the **dependent variable** and the second quantity the **independent variable**. In the above example, the distance traveled depends on the time, so the distance is the dependent variable and the time is the independent variable. We find the average rate of change as follows:

$$\text{average rate of change} = \frac{\text{change in dependent variable}}{\text{change in independent variable}}$$

The changes in the variables are found by subtracting. The order of subtraction does not matter in theory, but we'll follow the fairly standard convention

$$\text{average rate of change} = \frac{\text{final value minus intial value for dependent variable}}{\text{final value minus intial value for independent variable}}$$

This should remind you of the process for finding a slope. In fact, we will see soon that every average rate of change can be interpreted as the slope of a line.

The example on the next page will illustrate what we have just talked about.

◇ **Example 3.7(b):** Again you were driving, this time from Klamath Falls to Medford. At the top of the pass on Highway 140, which is at about 5000 feet elevation, the outside thermometer of your car registered a temperature of 28°F. In Medford, at 1400 feet of elevation, the temperature was 47°. What was the average rate of change of temperature with respect to elevation?

Solution: The words "temperature with respect to elevation" implies that temperature is the dependent variable and elevation is the independent variable. From the top of the pass to Medford we have

$$\text{average rate of change} = \frac{47°\text{F} - 28°\text{F}}{1400 \text{ feet} - 5000 \text{ feet}} = \frac{19°\text{F}}{-3600 \text{ feet}} = -0.0053°\text{F per foot}$$

How do we interpret the negative sign with our answer? Usually, when interpreting a rate of change, its value is *the change in the dependent variable for each INCREASE in one unit of the independent variable*. So the above result tells us that the temperature (on that particular day, at that time and in that place) *decreases* by 0.0053 degrees Fahrenheit for each foot of elevation *gained*.

Average Rate of Change of a Function

You probably recognized that what we have been calling independent and dependent variables are the "inputs" and "outputs" of a function. This leads us to the following:

Average Rate of Change of a Function

For a function $f(x)$ and two values a and b of x with $a < b$, the **average rate of change of f with respect to x** over the interval $[a, b]$ is

$$\left.\frac{\Delta f}{\Delta x}\right|_{[a,b]} = \frac{f(b) - f(a)}{b - a}$$

There is no standard notation for average rate of change, but it is standard to use the capital Greek letter delta (Δ) for "change." The notation $\left.\frac{\Delta f}{\Delta x}\right|_{[a,b]}$ could be understood by any mathematician as change in f over change in x, over the interval $[a, b]$. Because it is over an interval it must necessarily be an average, and the fact that it is one change divided by another indicates that it is a *rate* of change.

◇ **Example 3.7(c):** For the function $h(t) = -16t^2 + 48t$, find the average rate of change of h with respect to t from $t = 0$ to $t = 2.5$.

Solution:

$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8$$

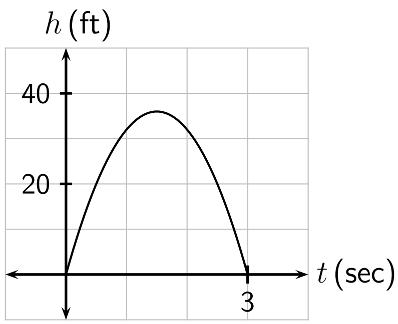
You might recognize $h(t) = -16t^2 + 48t$ as the height h of a projectile at any time t, with h in feet and t in seconds. The units for our answer are then feet per second, indicating that over the first 2.5 seconds of its flight, the height of the projectile increases at an average rate of 8 feet per second.

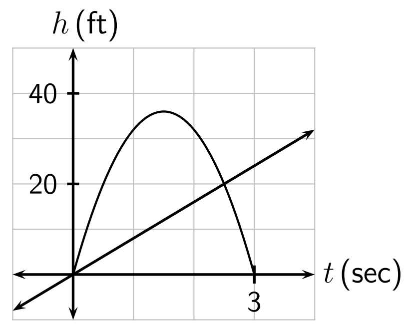
Secant Lines

The graph below and to the left is for the function $h(t) = -16t^2 + 48t$ of Example 3.7(c). To the right is the same graph, with a line drawn through the two points $(0, 0)$ and $(2.5, 20)$. Recall that the average rate of change in height with respect to time from $t = 0$ to $t = 2.5$ was determined by

$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8 \text{ feet per second}$$

With a little thought, one can see that the average rate of change is simply the slope of the line in the picture to the right below.

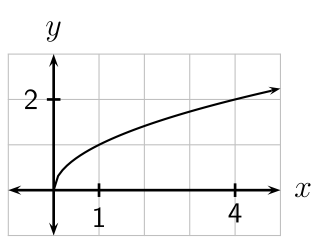


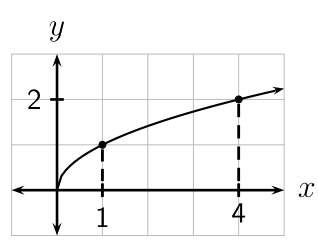


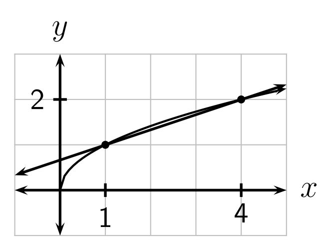
A line drawn through two points on the graph of a function is called a **secant line**, and the slope of a secant line represents the rate of change of the function between the values of the independent variable where the line intersects the graph of the function.

◇ **Example 3.7(d):** Sketch the secant line through the points on the graph of the function $y = \sqrt{x}$ where $x = 1$ and $x = 4$. Then compute the average rate of change of the function between those two x values.

Solution: The graph below and to the left is that of the function. In the second graph we can see how we find the two points on the graph that correspond to $x = 1$ and $x = 4$. The last graph shows the secant line drawn in through those two points.







The average rate of change is $\dfrac{2-1}{4-1} = \dfrac{1}{3}$, the slope of the secant line.

Difference Quotients

The following quantity is the basis of a large part of the subject of calculus:

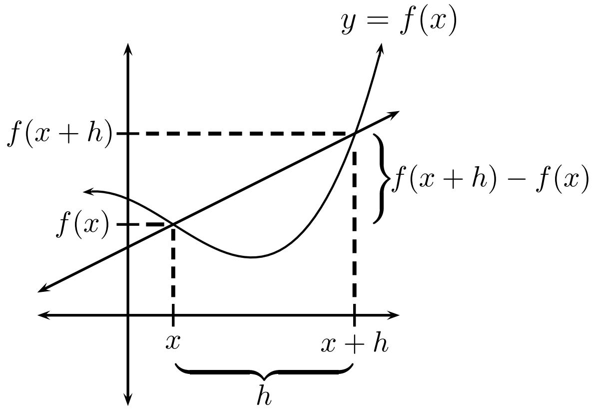
Difference Quotient for a Function

For a function $f(x)$, the **difference quotient** for f is

$$\frac{f(x+h)-f(x)}{h}$$

Let's rewrite the above expression to get a better understanding of what it means. With a little thought you should agree that

$$\frac{f(x+h)-f(x)}{h}=\frac{f(x+h)-f(x)}{(x+h)-a}$$



This looks a lot like a slope, and we can see from the picture to the right that it is. The quantity $f(x+h)-f(x)$ is a "rise" and $(x+h)-x=h$ is the corresponding "run." The difference quotient is then the slope of the tangent line through the two points $(x, f(x))$ and $(x+h, f(x+h))$.

◇ **Example 3.7(e):** For the function $f(x)=3x-5$, find and simplify the difference quotient. Another Example

$$\frac{f(x+h)-f(x)}{h}=\frac{[3(x+h)-5]-[3x-5]}{h}=\frac{3x+3h-5-3x+5}{h}=\frac{3h}{h}=3$$

Note that in the above example, the h in the denominator eventually canceled with one in the numerator. *This will always happen if you carry out all computations correctly!*

◇ **Example 3.7(f):** Find and simplify the difference quotient $\dfrac{g(x+h)-g(x)}{h}$ for the function $g(x)=3x-x^2$. Another Example Another Example

Solution: With more complicated difference quotients like this one, it is sometimes best to compute $g(x+h)$, then $g(x+h)-g(x)$, before computing the full difference quotient:

$$g(x+h)=3(x+h)-(x+h)^2=3x+3h-(x^2+2xh+h^2)=3x+3h-x^2-2xh-h^2$$

$$g(x+h) - g(x) = (3x + 3h - x^2 - 2xh - h^2) - (3x - x^2) = 3h - 2xh - h^2$$

$$\frac{g(x+h) - g(x)}{h} = \frac{3h - 2xh - h^2}{h} = \frac{h(3 - 2x - h)}{h} = 3 - 2x - h$$

Section 3.7 Exercises To Solutions

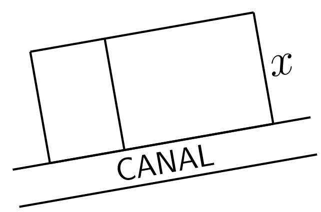
1. In Example 2.5(a) we examined the growth of a rectangle that started with a width of 5 inches and a length of 8 inches. At time zero the width started growing at a constant rate of 2 inches per minute, and the length began growing by 3 inches per minute. We found that the equation for the area A (in square inches) as a function of time t (in minutes) was $A = 6t^2 + 31t + 40$. Determine the average rate of increase in area with respect to time from time 5 minutes to time 15 minutes. Give units with your answer!

2. The height h (feet) of a projectile at time t (seconds) is given by $h = -16t^2 + 144t$.

   (a) Find the average rate of change in height, with respect to time, from 4 seconds to 7 seconds. Give your answer as a sentence that includes these two times, your answer, and whichever of the words *increasing* or *decreasing* that is appropriate.

   (b) Find the average rate of change in height, with respect to time, from time 2 seconds to time 7 seconds. Explain your result *in terms of the physical situation*.

3. A farmer is going to create a rectangular field with two compartments against a straight canal, as shown to the right, using 1000 feet of fence. No fence is needed along the side formed by the canal.



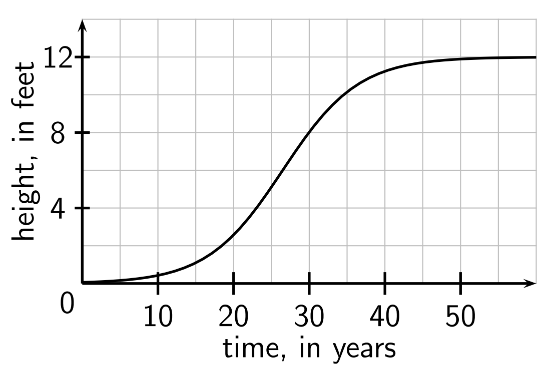
   (a) Find the total area of the field when x is 200 feet and again when x is 300 feet.

   (b) Find the average rate of change in area, with respect to x, between when x is 200 feet and when it is 300 feet. Give units with your answer.

4. The graph to the right shows the heights of a certain kind of tree as it grows. Use it to find the average rate of change of height with respect to time from

   (a) 10 years to 50 years

   (b) 25 years to 30 years



5. In a previous exercise you found the revenue equation $R = 20000p - 100p^2$ for sales of Widgets, where p is the price of a widget and R is the revenue obtained at that price.

   (a) Find the average change in revenue, with respect to price, from a price of \$50 to \$110. Include units with your answer.

   (b) Sketch a graph of the revenue function and draw in the secant line whose slope represents your answer to (a). Label the relevant values on the horizontal and vertical axes.

6. (a) Sketch the graph of the function $y = 2^x$. Then draw in the secant line whose slope represents the average rate of change of the function from $x = -2$ to $x = 1$.

   (b) Determine the average rate of change of the function from $x = -2$ to $x = 1$. Give your answer in (reduced) fraction form.

7. Find the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$.

8. Find the average rate of change of $g(x) = \dfrac{3}{x-2}$ from $x = 4$ to $x = 7$.

9. Consider the function $h(x) = \frac{2}{3}x - 1$.

   (a) Find the average rate of change from $x = -6$ to $x = 0$.

   (b) Find the average rate of change from $x = 1$ to $x = 8$.

   (c) You should notice two things about your answers to (a) and (b). What are they?

10. Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for each of the following.

    (a) $f(x) = x^2 - 3x + 5$

    (b) $f(x) = 3x^2 + 5x - 1$

    (c) $f(x) = 3x - 1$

    (d) $f(x) = x^2 + 7x$

    (e) $f(x) = 3x - x^2$

    (f) $f(x) = -\frac{3}{4}x + 2$

    (g) $f(x) = x^2 + 4x$

    (h) $f(x) = 7x^2 + 4$

11. For the function $f(x) = 3x^2$, find and simplify the difference quotient.

12. (a) Compute and simplify $(x+h)^3$

    (b) For the function $f(x) = x^3 - 5x + 1$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

    (c) In Exercise 7 you found the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$. That is equivalent to computing the difference quotient with $x = 1$ and $x + h = 4$. Determine the value of h, then substitute it and $x = 1$ into your answer to part (b) of this exercise. You should get the same thing as you did for Exercise 7!

13. For this exercise you will be working with the function $f(x) = \dfrac{1}{x}$.

   (a) Find and simplify $f(x+h) - f(x)$ by carefully obtaining a common denominator and combining the two fractions.

   (b) Using the fact that dividing by h is the same as multiplying by $\frac{1}{h}$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

3.8 Chapter 3 Exercises

To Solutions

1. Solve each of the following linear inequalities.

   (a) $8 - 5x \leq -2$

   (b) $6x - 3 < 63$

   (c) $3(x-2) + 7 < 2(x+5)$

   (d) $-4x + 3 < -2x - 9$

   (e) $8 - 5(x+1) \leq 4$

   (f) $7 - 4(3x+1) \geq 2x - 5$

2. Solve each of the following inequalities. (**Hint:** Get zero on one side first, if it isn't already.) Give your answers as inequalities, OR using interval notation.

   (a) $21 + 4x \geq x^2$

   (b) $y^2 + 3y \geq 18$

   (c) $8x^2 < 16x$

   (d) $\dfrac{1}{15}x^2 \leq \dfrac{1}{6}x - \dfrac{1}{10}$

3. Consider the function $f(x) = \frac{1}{4}(x+1)(x-3)$.

   (a) Sketch the graph of the function *without calculating anything*, just using the equation of the function as given. You should be able to locate exactly two points on the parabola.

   (b) Based on your graph, will the rate of change of the function from $x = -3$ to $x = 2$ be negative, or will it be positive? Add something to your graph to help you answer this question, and provide a few sentences of explanation.

4. **To do this exercise you need to have done Exercise 13 of the Chapter 1 Exercises.** In this section we found out how to determine the equation of a parabola from two points on the parabola, *where one of the points is the vertex*. If we don't know the vertex, we must have three points on the parabola in order to determine its equation. This exercise will lead us through the process for doing that. We will find the equation of the parabola containing the points $(-3, 3)$, $(1, 3)$, and $(3, -3)$.

   (a) Put the coordinates $(-3, 3)$ into the standard form $y = ax^2 + bx + c$ of a parabola, and rearrange to get something like $2a + 5b - c = -4$.

   (b) Repeat (a) for the other two points on the parabola.

   (c) You now have a system of three equations in three unknowns. Solve the system in the manner described in Exercise 13 of the Chapter 1 Exercises.

   (d) You have now found a, b and c, so you can write the equation $y = ax^2 + bx + c$ of the parabola, so do so. Then graph the equation using technology and check to see that it does indeed pass through the given points.

5. Find the equation of the parabola containing the points $(0, 8)$, $(1, 3)$ and $(4, 0)$. Check your answer by graphing with technology.

4 Polynomial and Rational Functions

Outcome/Performance Criteria:

4. Understand polynomial and rational functions.

   (a) Identify the degree, lead coefficient and constant term of a polynomial function from its equation.

   (b) Given the graph of a polynomial function, determine its possible degrees and the signs of its lead coefficient and constant term. Given the degree of a polynomial function and the signs of its lead coefficient and constant term, sketch a possible graph of the function.

   (c) Give the end behavior of a polynomial function, from either its equation or its graph, using "as $x \to a$, $f(x) \to b$ notation.

   (d) Graph a polynomial function from the factored form of its equation; given the graph of a polynomial function with its x-intercepts and one other point, give the equation of the polynomial function.

   (e) Solve a polynomial inequality.

   (f) Give the x- and y-intercepts and the equations of the vertical and horizontal asymptotes of a rational function from either the equation or the graph of the function.

   (g) Graph a rational function from its equation, without using a calculator.

   (h) Give the end behavior and behavior of a rational function from its graph, using "as $x \to a$, $f(x) \to b$ notation, where a is a real number, $-\infty$ or ∞.

4.1 Introduction to Polynomial Functions

Performance Criteria:

4. (a) Identify the degree, lead coefficient and constant term of a polynomial function from its equation.

   (b) Given the graph of a polynomial function, determine its possible degrees and the signs of its lead coefficient and constant term. Given the degree of a polynomial function and the signs of its lead coefficient and constant term, sketch a possible graph of the function.

   (c) Give the end behavior of a polynomial function, from either its equation or its graph, using "as $x \to a$, $f(x) \to b$ notation.

A polynomial function consists of various whole number (0, 1, 2, 3, ...) powers of x, each possibly multiplied by some number, all added (or subtracted) together. Here are some examples of polynomial functions:

$$P(x) = -7x^{14} + 2x^{11} + x^6 - \frac{2}{3}x^2 - 3 \qquad g(x) = x^2 - x - 6$$

$$y = 5x - 9 \qquad h(x) = 2 + 3x - 5x^2 + 16x^5$$

These *are not* polynomial functions:

$$f(x) = \sqrt{x^3 - 5x^2 + 7x - 1} \qquad y = 7^x \qquad R(x) = \frac{3x^2 - 5x + 2}{x^2 - 9}$$

The last function above is a fraction made up of two polynomials. It is not a polynomial function, but is a type of function called a **rational function**, which is the ratio of two polynomials. We'll work with rational functions in Sections 4.3 and 4.4. Let's make a few observations about polynomial functions:

- Most polynomial functions include a number that appears not to be multiplying a power of x, but we could say that such a number is multiplying x^0, since $x^0 = 1$.
- The powers of x can be ordered in any way, although we will generally order them from highest to lowest, as in P above.
- Linear and quadratic functions are just special cases of polynomial functions.
- A polynomial function can be evaluated for any value of x, so the domain of any polynomial function is all real numbers.

As with the other functions we have studied, we are interested in the behaviors of polynomial functions and how the graphs of such functions are related to their equations.

Vocabulary of Polynomial Functions

There is some language that we use when discussing both the equations and graphs of polynomial functions. Let's first address the equations.

- The numbers multiplying the powers of x in a polynomial function are called **coefficients** of the polynomial. For example, the coefficients of $f(x) = 5x^4 - 7x^3 + 3x - 4$ are 5, -7, 0, 3 and -4. Note that the signs in front of any of the coefficients are actually part of the coefficients themselves.

- Each coefficient and its power of x together are called **terms** of the polynomial. The terms of this polynomial are $5x^4, -7x^3, 3x$ and -4.

- The coefficient of the highest power of x is called the **lead coefficient** of the polynomial, and if there is a number without a power of x, it is called the **constant coefficient** or **constant term** of the polynomial. For our example of $f(x) = 5x^4 - 7x^3 + 3x - 4$, the lead coefficient is 5 and the constant coefficient is -4. We will see that both the lead coefficient and the constant coefficient give us valuable information about the behavior of a polynomial.

- We will also find that the exponent of the highest power of x gives us some information as well, so we have a name for that number. It is called the **degree** of the polynomial. The polynomial $f(x) = 5x^4 - 7x^3 + 3x - 4$ has degree 4. (We also say this as "f is a fourth degree polynomial.") *Note that the degree is simply a whole number!*

- Let us make special note of polynomials of degrees zero, one, and two. A polynomial of degree zero is something like $f(x) = -2$, which we also call a **constant function**. The term *constant* is used because the output $f(x)$ remains the same (in this case -2) no matter what the input value x is. An example of a first degree polynomial would be a linear function $f(x) = \frac{2}{3}x + 1$, which we have already studied. You should know that when we graph it we will get a line with a slope of $\frac{2}{3}$ and a y-intercept of 1. A second degree polynomial like $f(x) = 2 + 5x - 4x^2$ is just a quadratic function, and its graph is of course a parabola (in this case, opening down).

◇ **Example 4.1(a):** For the polynomial function $P(x) = -7x^{14} + 2x^{11} + x^6 - \frac{2}{3}x^2 - 3$, give

(a) the terms of the polynomial,

(b) the coefficients of the polynomial,

(c) the lead coefficient of the polynomial,

(d) the constant term of the polynomial,

(e) the degree of the polynomial.

Solution:

(a) The terms of the polynomial are $-7x^{14}$, $2x^{11}$, x^6, $-\frac{2}{3}x^2$ and -3.

(b) The coefficients of the polynomial are -7, 2, 1, $-\frac{2}{3}$ and -3.

(c) The lead coefficient of the polynomial is -7.

(d) The constant term of the polynomial is -3.

(e) The degree of the polynomial is 14.

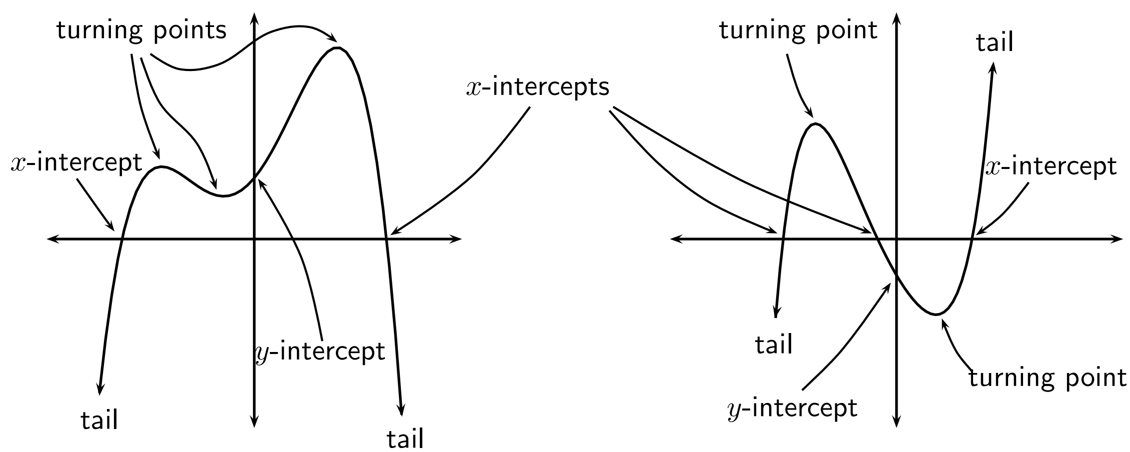
At times we will want to discuss a general polynomial, without specifying the degree or coefficients. To do that we write the general polynomial as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \ .$$

The symbols $a_n, a_{n-1}, ..., a_2, a_1$ represent the coefficients of $x^n, x^{n-1}, ..., x^2$ and x, and a_0 is (always) the constant coefficient. If, for example, a polynomial has no x^7 term, then $a_7 = 0$. Finally, let us mention that we sometimes use $P(x)$ instead of $f(x)$ to denote a polynomial function.

Graphs of Polynomial Functions

There is also some vocabulary associated with the graphs of polynomial functions. The graphs below are for two polynomial functions; we will use them to illustrate some terminology associated with the graph of a polynomial.

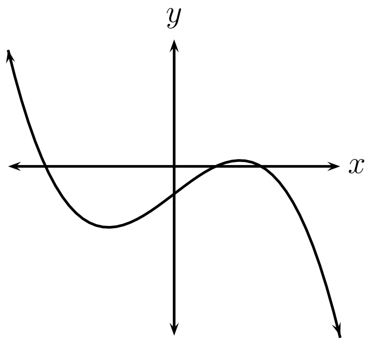


- Your eye may first be drawn to the "humps" of each graph (both "right side up" and "upside down"). The top of a "right side up" hump or the bottom of an "upside down" hump are called **turning points** of the graph. Note that turning points are also locations of maxima and minima (sometimes relative, sometimes absolute).

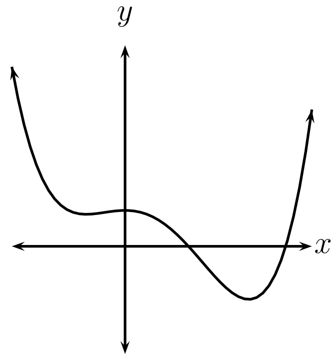
- The graph of any polynomial function always has exactly one y-intercept. The graph of a polynomial function can have various numbers of x-intercepts - we will investigate this in the exercises.

- The two ends of the graph are generally drawn with arrowheads to indicate that the graph keeps going. These ends are called "tails" of the graph. Note that graphs of polynomials spread forever to the left and right, since the domain of a polynomial function is all real numbers.

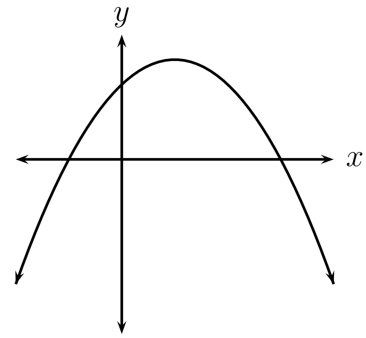
We'll now look at some polynomial functions and their graphs, in order to try to see how the graph of a polynomial function is related to the degree of the polynomial and to its coefficients. Six polynomial functions and their graphs are shown at the top of the next page.



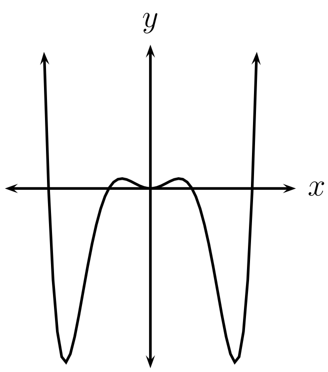
$f(x) = -2x^3 + 15x - 13$



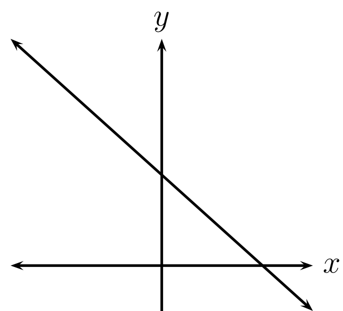
$P(x) = x^4 - 2x^3 - 3x^2 + 10$



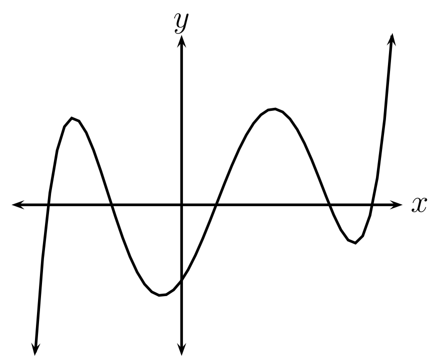
$y = -x^2 + 2x + 3$



$g(x) = x^6 - 7x^4 + 6x^2$



$y = -\frac{3}{2}x + 4$



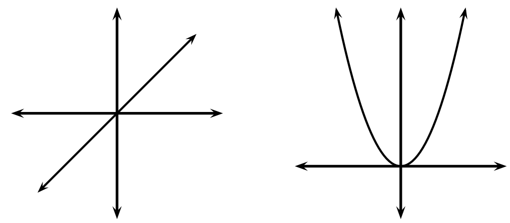
$$Q(x) = x^5 - 5x^4 - 23x^3 + 93x^2 + 130x - 200$$

The general shape of the graph of a polynomial function is dictated by its degree and the sign of its lead coefficient, and the vertical positioning is determined by the constant term. Study the graphs just given to see how they illustrate the following.

Graphs of Polynomial Functions:

- If the degree of the polynomial is even, both tails either go upward or both go downward. If the degree is odd, one tail goes up and the other goes down.
- If the lead coefficient of an even degree polynomial is positive, both tails go up; if the lead coefficient is negative, both tails go down.
- If the lead coefficient of an odd degree polynomial is positive, the left tail goes down and the right tail goes up. If the lead coefficient is negative, the left tail goes up and the right tail goes down.
- The number of turning points is *at most* one less than the degree of the polynomial. The number of x-intercepts is at most the degree of the polynomial.
- The y-intercept is the constant term.

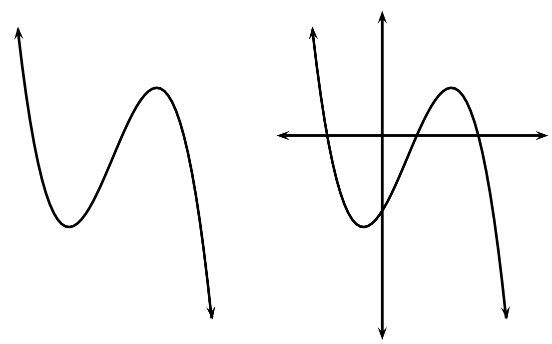
These things can be remembered by thinking of the odd and even functions whose graphs you are most familiar with, $y = x^1$ and $y = x^2$, shown to the right. $y = x^1$ has an odd degree and positive lead coefficient, and you can see that its left tail goes down and right tail up, as is the case for all polynomials with odd degree and positive lead coefficient. $y = x^2$ has even degree and positive lead coefficient and, like all such polynomials, both tails go up. If we are graphing something like $P(x) = -4x^3 + 7x^2 - 3x + 2$ we simply think of it like $y = -4x$, which slopes downward left-to-right, so its left tail goes up and its right tail down. The tails of $P(x)$ then do the same thing. Similarly, $f(x) = -5x^4 + 7x^3 - 2x + 1$ will have tails like $y = -5x^2$, with both tails going down.



$y = x$ $\qquad$ $y = x^2$

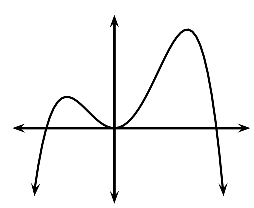
◇ **Example 4.1(b):** Sketch the graph of a polynomial function with degree three, negative lead coefficient and negative constant term.

Solution: Because the degree is odd, one tail goes up and the other goes down and, because of the negative lead coefficient the left tail goes up and the right tail goes down. Therefore we know the graph looks something like the first one shown to the right. We then simply need to put in coordinate axes in such a way that the y-intercept is negative, as shown in the second picture. We know this must be the case because the constant term is negative.

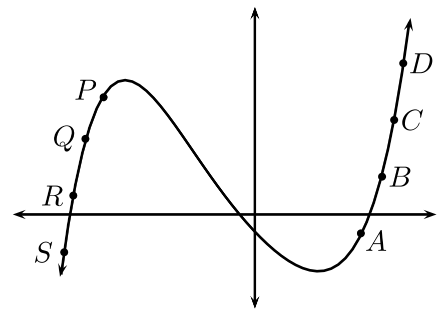


◇ **Example 4.1(c):** Give the smallest possible degree, the sign of the lead coefficient, and the sign of the constant term for the polynomial function whose graph is shown below and to the right.

Solution: Because both tails go the same way the degree must be even. To have three turning points the degree must be at least four. For both tails to go down the lead coefficient must be negative. The y-intercept is zero, so the constant term must be zero also (or we could just say there is no constant term).



End Behavior of Polynomial Functions



Given the graph of a function on a coordinate grid, we can get reasonably good values of $f(x)$ for any value of x, and vice versa. Our goal now is to give some sort of written description of the function's behavior at the edges of the graph and beyond. We'll use the graph to the right to explain how this is done.

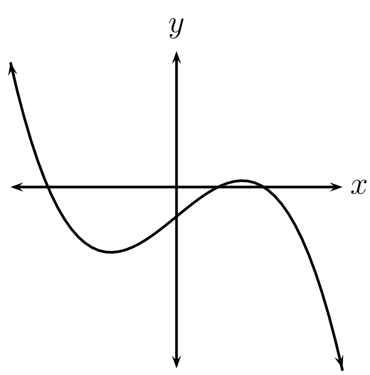
Consider the points A, B, C and D. Note that as we progress from one point to the next, in that order, the values of x are getting larger and larger. We will indicate this by the notation $x \to \infty$. That is, there is a place we call "infinity" at the right end of the x-axis, and the values of x are headed toward it. At the same time the corresponding values of y are also getting larger as we proceed from point A to point D. We summarize all this by writing

$$\text{As } x \to \infty, \ y \to \infty.$$

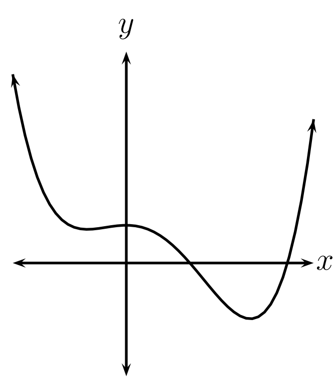
Similarly, as we proceed from point P to Q, R and S, the values of x are getting smaller and smaller, and the corresponding y values are as well. In this case we write

$$\text{As } x \to -\infty, \ y \to -\infty.$$

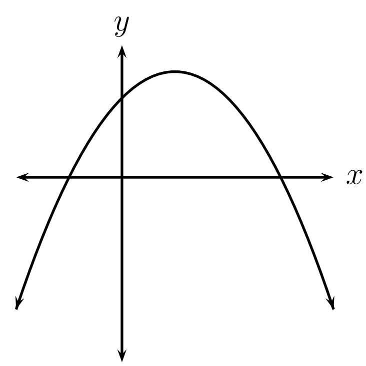
◇ **Example 4.1(d):** Describe the end behavior of each of the polynomial functions graphed below using "as $x \to a$, $y \to b$" statements.



$f(x) = -2x^3 + 15x - 13$



$P(x) = x^4 - 2x^3 - 3x^2 + 10$



$y = -x^2 + 2x + 3$

Solution: For the first function, f, we have

$$\text{as } x \to -\infty, \ y \to \infty \ \text{ and as } x \to \infty, \ y \to -\infty.$$

For P, the second function,

$$\text{as } x \to -\infty, \ y \to \infty \ \text{ and as } x \to \infty, \ y \to \infty.$$

Finally, for the last function,

$$\text{as } x \to -\infty, \ y \to -\infty \ \text{ and as } x \to \infty, \ y \to -\infty.$$

Section 4.1 Exercises   To Solutions

1. For the polynomial $P(x) = -5x^4 + 3x^2 - 7x + 2$, give the

   (a) degree   (b) coefficients   (c) terms

   (d) lead coefficient   (e) constant term

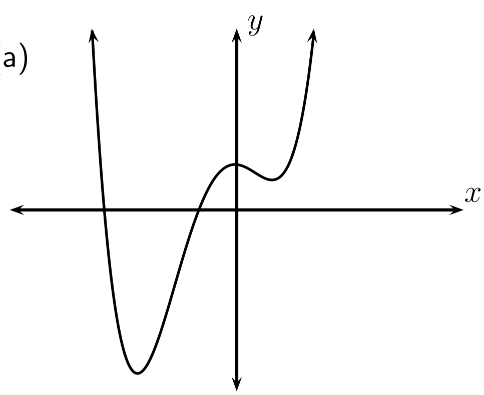
2. For each of the following polynomial functions, give the degree, lead coefficient, and the constant term.

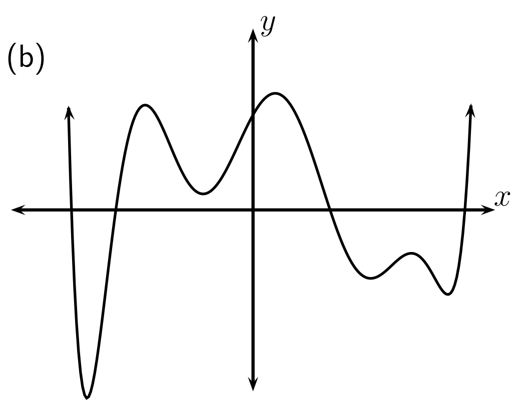
   (a) $y = -3x^4 + 5x^2 - 1$ (b) $f(x) = 7x^3 - 2x^2 + 5x + 3$

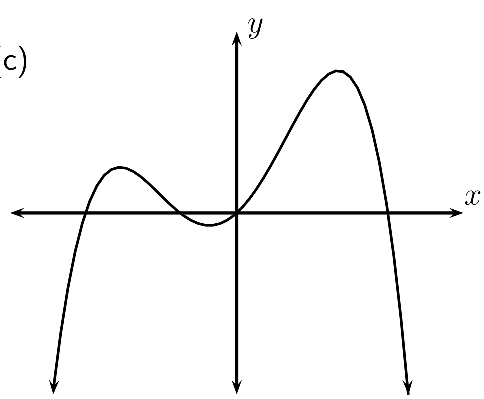
   (c) $g(x) = x^2 - 9$ (d) $P(x) = x^5 - 2x^4 + 10x^3 + 3x^2 - 5x$

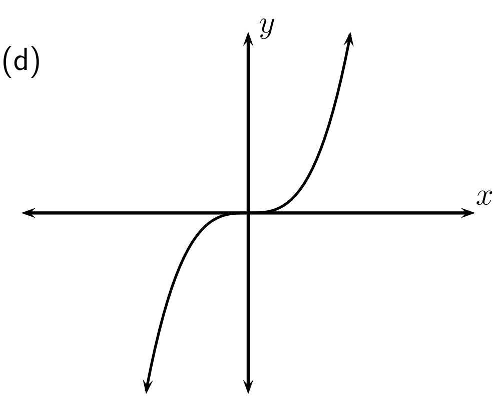
   (e) $y = \frac{2}{3}x - 5$ (f) $h(x) = 10$

3. For each of the following graphs of polynomial functions, tell what you can about the lead coefficient, the constant coefficient, and the degree of the polynomial.

(a)


(b)


(c)


(d)


4. (a) - (d) For each graph in Exercise 1, give the end behaviors in the manner used in Example 4.1(d).

5. Sketch the graph of a polynomial function having the following characteristics. Some are not possible; in those cases, write "not possible".

   (a) Degree three, negative lead coefficient, one x-intercept.

   (b) Degree three, negative lead coefficient, two x-intercepts.

   (c) Degree three, positive lead coefficient, four x-intercepts.

   (d) Degree three, negative lead coefficient, no x-intercepts.

   (e) Degree four, positive lead coefficient, three x-intercepts.

   (f) Degree four, positive lead coefficient, one x-intercept.

   (g) Degree four, negative lead coefficient, two x-intercepts.

6. For what degrees is it possible for a polynomial to not have any x-intercepts?

7. A polynomial has exactly 3 turning points. Which of the following are possible numbers of x-intercepts of the polynomial: 0, 1, 2, 3, 4, 5?

8. A polynomial has exactly three x-intercepts. Which of the following are possible numbers of turning points for the polynomial: 0, 1, 2, 3, 4, 5?

9. How many x-intercepts does a polynomial of degree one have?

10. How many x-intercepts does a polynomial of degree zero have?

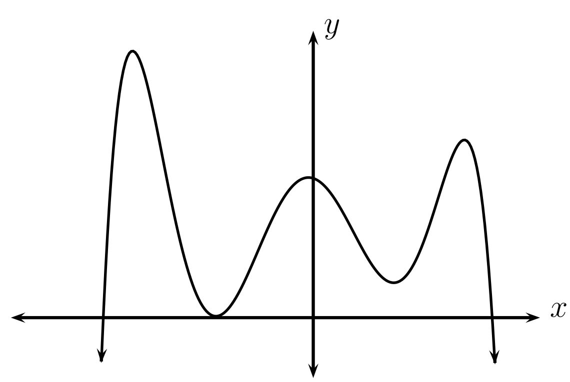
11. Which of the polynomials below could have the graph shown to the right? For any that couldn't, tell why.

   (a) $f(x) = 5x^6 - 7x^3 + x + 3$

   (b) $f(x) = -5x^6 - 7x^3 + x + 3$

   (c) $f(x) = -5x^6 - 7x^3 + x - 3$

   (d) $f(x) = -5x^4 - 7x^3 + x + 3$



12. (a) Does $f(x) = -5x^3 + 7x^2 - 3x + 1$ have an absolute maximum? Absolute minimum? Both? Neither? Explain.

   (b) How about $g(x) = x^4 + 7x^2 - 3$? Again, explain.

   (c) Discuss the existence of relative and absolute minima and maxima for even degree polynomial functions. Do the same for odd degree polynomial functions.

13. Give the general form of each of the following:

    (a) The domain of an odd degree polynomial function.

    (b) The range of an odd degree polynomial function.

    (c) The domain of an even degree polynomial function.

    (d) The range of an even degree polynomial function.

14. Find the equation of the secant line for $y = -2x^3 + 7x - 1$ from $x = -2$ to $x = 1$. Give your answer in exact form, so no rounded decimal values.

15. Graph the polynomial function $y = -\frac{1}{5}x^5 - 2x^4 - 5x^3$ using *Desmos* or some other technology. Adjust your window so that you can see all features of interest on the graph. Note that if you use *Desmos*, clicking on any high or low points will show their coordinates.

    (a) Give the interval(s) on which the function is increasing.

    (b) Give the interval(s) on which the function is positive.

    (c) Describe all minima and/or maxima of the function. (Tell whether they are relative or absolute, what their values are, and where they occur.)

4.2 Polynomial Functions in Factored Form

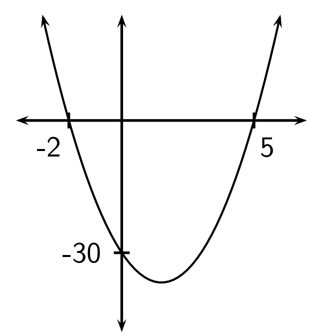
Performance Criteria:

4. (d) Graph a polynomial function from the factored form of its equation; given the graph of a polynomial function with its x-intercepts and one other point, give the equation of the polynomial function.

Let's once again begin with an example.

◇ **Example 4.2(a):** Find the intercepts of $f(x) = 3x^2 - 9x - 30$ and sketch the graph using only those points.

Solution: First we note that the graph is a parabola opening upward, and can see that the y-intercept is -30. Next we factor the right side to get



$$f(x) = 3x^2 - 9x - 30 = 3(x^2 - 3x - 10) = 3(x + 2)(x - 5).$$

From this we can easily see that $x = -2$ and $x = 5$ are the x-intercepts of the function. It was easy to see from the original that $f(0) = -30$, but it isn't too hard to get that from the factored form:

$$f(0) = 3(0 + 2)(0 - 5) = 3(2)(-5) = -30.$$

We now plot the intercepts and draw the graph, shown above and to the right.

In the previous section we dealt with polynomial functions in the **standard form**

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\,.$$

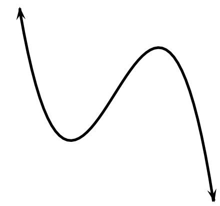
Many such functions can be factored into the **factored form**

$$P(x) = a_n(x - x_1)(x - x_2)(x - x_3)\cdots(x - x_n)\,,$$

where x_1, x_2, ..., x_n are real numbers and a_n is the lead coefficient regardless of which form we are talking about. In this section we will see how to get the graph of a polynomial function from its factored form, and vice versa, using a method just like that used in the example above.

◇ **Example 4.2(b):** Sketch a graph of the function $f(x) = -\frac{1}{2}(x+1)(x-1)(x-4)$, indicating clearly the intercepts.

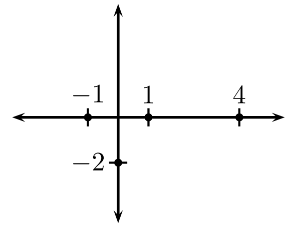
Solution: If we were to imagine multiplying the right side of this equation out, the first term of the result would be $-\frac{1}{2}x^3$, so we would expect a graph with the left tail going up and the right tail going down. We would also expect our graph to probably have two turning points. See the diagram to the right for what we would expect the graph to look like.



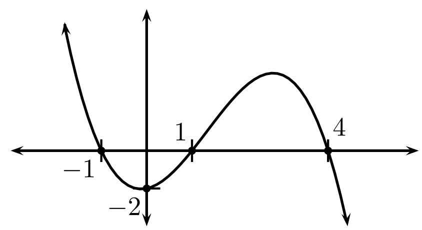
Next we would observe that the graph has x-intercepts -1, 1 and 4. (These are obtained by noting that these values of x result in $f(x) = 0$.) If we then let $x = 0$ and compute $f(0)$ we get

$$f(0) = -\tfrac{1}{2}(1)(-1)(-4) = -2,$$

which is the y-intercept of our graph. The graph to the right shows the intercepts plotted on a set of coordinate axes.



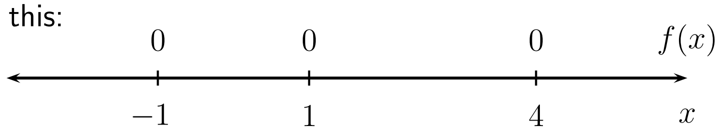
Now we just have to sketch a graph that looks like what we expected, and passing through the intercepts we have found. The final graph is shown to the right.



We can instead determine the sign (positive or negative) of the function in each interval determined by the x-intercepts to graph a polynomial function. The next example shows this method for the same function as graphed in the previous example.

◇ **Example 4.2(c):** Determine the sign of $f(x) = -\frac{1}{2}(x+1)(x-1)(x-4)$ in each interval determined by the x-intercepts, and use that information to graph the function.

Solution: We know that $f(x) = 0$ when $x = -1,\ 1,\ 4,$ so we draw and label a number line indicating this:



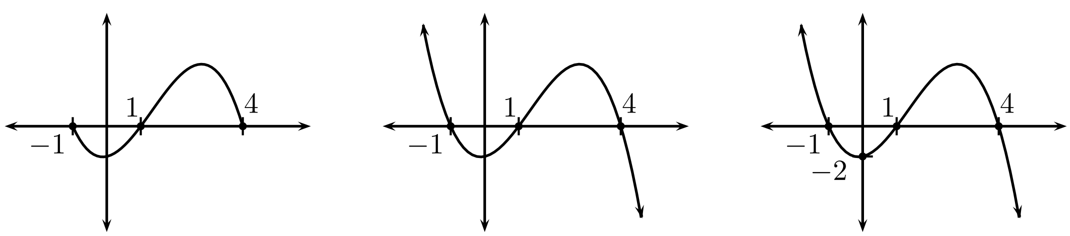
Note that the intercepts divide the x-values into the four intervals $(-\infty, -1)$, $(-1, 1)$, $(1, 4)$ and $(4, \infty)$. We then pick a number in each of those intervals and determine the sign of the function for that number - that will be the sign throughout the entire interval. For example, when $x = 2$, $-\frac{1}{2}$ is negative, $x+1$ is positive, $x-1$ is positive and $x-4$ is negative, so the sign of $f(2)$ is $(-)(+)(+)(-) = +$. This is shown in the second row of the table at the top of the next page. The top row shows the factors of the function f and each row give the signs of each factor for the value of x given at the left end of the row, along with the product of all of them. In each row below that we show the sign of the factor at the top of the column and, in the last column, the sign of the product.

	$-\frac{1}{2}$	$(x+1)$	$(x-1)$	$(x-4)$		$-\frac{1}{2}(x+1)(x-1)(x-4)$
$x=2:$	$(-)$	$(+)$	$(+)$	$(-)$	$=$	$+$
$x=-2:$	$(-)$	$(-)$	$(-)$	$(-)$	$=$	$+$
$x=0:$	$(-)$	$(+)$	$(-)$	$(-)$	$=$	$-$
$x=5:$	$(-)$	$(+)$	$(+)$	$(+)$	$=$	$-$

After making such a table we can then put the signs of f in each of the intervals between the x-intercepts:

$+$	0	$-$	0	$+$	0	$-$	$f(x)$
	-1		1		4		x

The above sign chart indicates that the appearance of the graph of the function between -1 and 4 is as shown below and to the left. Our understanding of the behaviors of the tails of a function then allows us to add tails as shown in the middle below. Finally, we can evaluate the function for $x = 0$ to determine that the y-intercept is -2, which is indicated by the final graph shown below and to the right.



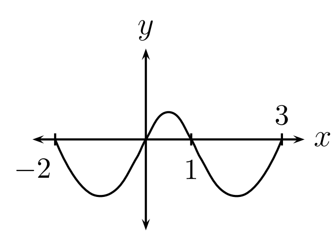
A number line with locations of x-intercepts of a function and signs of the function between them is what is often referred to as a **sign chart**.

◇ **Example 4.2(d):** The sign chart for $g(x) = \frac{1}{3}x(x+2)(x-1)(x-3)$ is

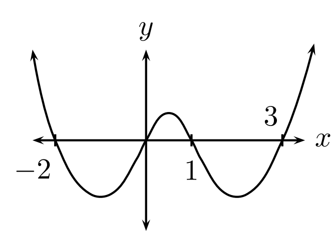
$+$	0	$-$	0	$+$	0	$-$	0	$+$	$g(x)$
	-2		0		1		3		x

Use this sign chart to sketch the graph of the function.

Solution: The $-$ sign between -2 and 0 tells us that the graph "loops" below the x-axis there. It loops above the x-axis from 0 to 1 and below again from 1 to 3. At this point we know the graph looks like figure (a) to the right



To the left of -2 and to the right of 3 the graph is above the x-axis. From our understanding of graphs of polynomial functions, we would guess the tails are like figure (b) to the right.



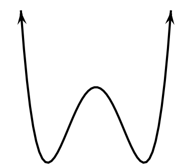
Note that before making the sign chart for the above function we could have determined that it is a fourth degree polynomial with positive lead coefficient, so both tails must go up and it could have as many as three humps. We could also get the y-intercept from

$$g(0) = \frac{1}{3}(0)(0+2)(0-1)(0-3) = 0$$

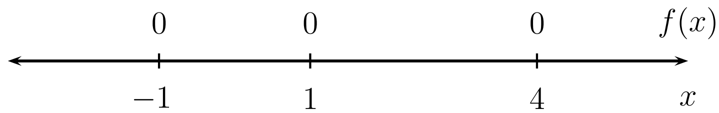
Of course we already knew the graph went through the origin, but if the y-intercept was not zero, we could determine it this way.

The next example illustrates an important variation from what we have seen so far.

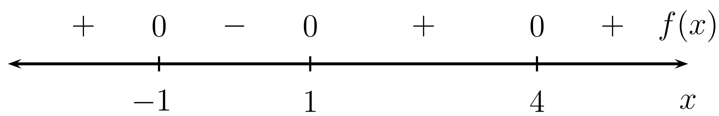
◇ **Example 4.2(e):** Use a sign chart to sketch the graph of $f(x) = \frac{1}{4}(x+1)(x-1)(x-4)^2$.



Solution: First we note that the polynomial is fourth degree with a positive lead coefficient, so the graph will look something like the one to the right. We know that $f(x) = 0$ at $x = -1,\ 1,\ 4.$ so our sign chart starts like this:



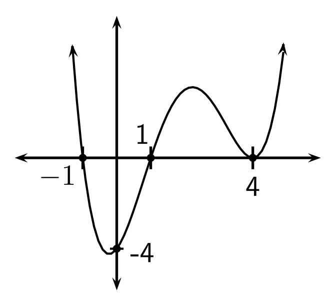
When $x = -2$, $\frac{1}{4} > 0$, $x+1 < 0$, $x-1 < 0$ and $(x-4)^2 > 0$, so the sign of $f(x)$ is $(+)(-)(-)(+) = +$. When $x = 0$, the sign of $f(x)$ is $(+)(+)(-)(+) = -$. Checking the signs for x values in the remaining two intervals, we end up with this sign chart:



This sign chart indicates that y is zero at $x = 4$, and on either side of 4 the y values are positive. This means that the lower right hand turning point in our original sketch of the graph has its bottom at the point $(4, 0)$. Because the y values are negative between -1 and 1, the lower left turning point is below the x-axis. Let's find the y-intercept:

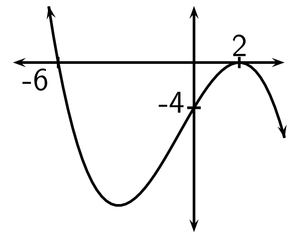
$$f(0) = \tfrac{1}{4}(0+1)(0-1)(0-4)^2 = \tfrac{1}{4}(-16) = -4$$

The final graph is shown to the right, with the intercepts labeled.



The above example illustrates that when our polynomial function has a factor like $(x-3)^2$ (or with any other even power) the graph of the function will "bounce off" of the x-axis at the x-intercept of 3. The next example will use this idea in showing how we obtain the factored equation of a polynomial whose graph is given.

◇ **Example 4.2(f):** Give the equation of a polynomial function (in factored form) that would have a graph like the one shown to the right.



Solution: We will create the equation of the function in a progressive manner, using three steps. Before beginning, though, we should note that the polynomial should be third degree, with a negative lead coefficient. Because of the locations of the intercepts, our first guess for the equation would be something like

$$y = (x + 6)(x - 2).$$

This is a second degree polynomial, but we note that because the graph bounces off of the x-axis at $x = 2$. Therefore the equation should really look like

$$y = (x + 6)(x - 2)^2,$$

which is now a third degree polynomial. There are still two problems with this equation, though: the lead coefficient is not negative, which it needs to be, and the y-intercept is $y = (6)(-2)^2 = 24$. We note that the factored form usually contains a lead coefficient factor a:

$$y = a(x + 6)(x - 2)^2.$$

We have not yet used the y-intercept, so it is likely the key to finding the value of a. Indeed, the y-intercept is really the point $(0, -4)$, so we can substitute those values into our equation and solve for a:

$$\begin{aligned} -4 &= a(0 + 6)(0 - 2)^2 \\ -4 &= 24a \\ \tfrac{-4}{24} &= a \\ -\tfrac{1}{6} &= a \end{aligned}$$

Therefore the factored equation of the polynomial with the given graph is $y = -\frac{1}{6}(x + 6)(x - 2)^2$.

Section 4.2 Exercises To Solutions

1. Sketch the graph of $f(x) = x(x + 3)(x - 2)$, indicating all the intercepts clearly. Check yourself with your calculator or other technology.

2. Sketch the graph of $g(x) = (x + 1)(x + 3)^2$, indicating all the intercepts clearly. Check yourself with your calculator or other technology.

3. Consider the function $h(x) = \frac{1}{4}(x+1)(x-1)(x-4)^3$, which is very similar to the function f from Example 4.2(e).

   (a) Try to graph this function "by hand." Check your answer with some technology.

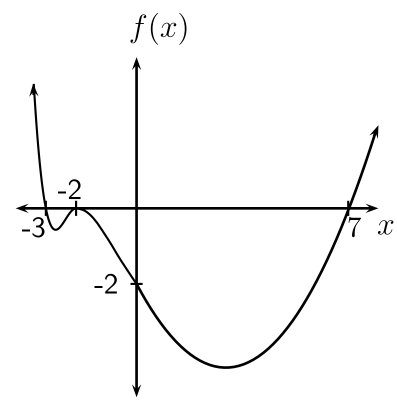
   (b) Both f and h have x-intercepts at $x = 4$, but their behaviors are different there. Describe the difference.

4. Consider the function whose graph is shown below and to the right.

   (a) What are the factors of the polynomial having the graph shown?

   (b) One of your factors from (a) should be squared - which one is it?

   (c) Create the equation of a polynomial function by multiplying your factors from (a) together, with a squared on the term that should have it. (**DO NOT** multiply it out!) What are the degree and lead coefficient of your polynomial? Based on those, should the graph of your equation look something like the one given?

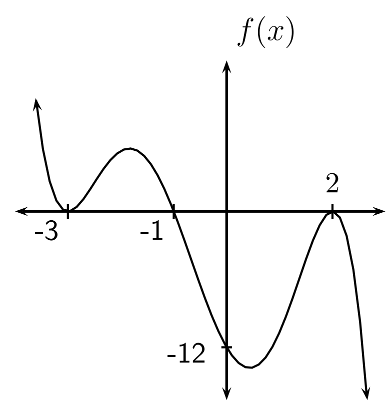


   (d) Find the y-intercept for your equation. It should *NOT* match the graph.

   (e) Here's how to make your equation have the correct y-intercept: Put a factor of a on the front of your polynomial function from (c), then substitute in x and y values for the y-intercept and solve for a.

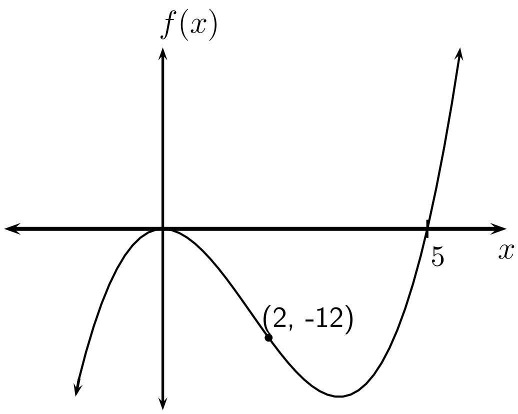
   (f) What is the equation of the polynomial function that will have a graph like the one shown?

5. Use a process like what you did in Exercise 4 to determine the factored form of a polynomial function whose graph is the one shown to the right. Check yourself with your calculator. The graph will not look quite like this one, but it should have the same shape and intercepts.

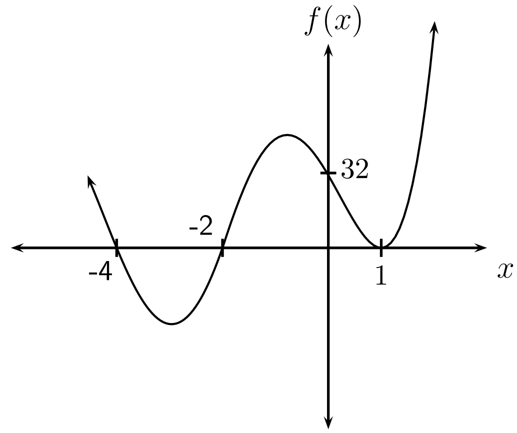


6. (a) Find the factored form of a polynomial function that gives the graph shown below and to the left.

   (b) Find the factored form of a polynomial function that gives the graph shown below and to the right.



Exercise 6(a)



Exercise 6(b)

7. Find the graph of each of the following, without using your calculator or other technology. Then check yourself using technology. Your graph may not look quite like the graph, but it should have the same shape and intercepts. *A couple of them you will have to factor.*

   (a) $f(x) = \frac{1}{2}(x-1)^2(x+2)$

   (b) $f(x) = x^3 - x^2 - 2x$ (Hint: Common factor.)

   (c) $f(x) = -\frac{1}{6}(x-1)(x+2)(x-3)(x+4)$

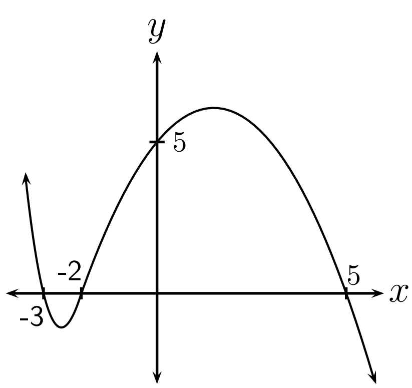
   (d) $f(x) = -\frac{1}{4}x^4 + 2x^2 - 4$ (**Hint:** First factor out $-\frac{1}{4}$, then factor like a quadratic.)

   (e) $f(x) = -\frac{1}{3}(x-1)(x+3)(x-3)$

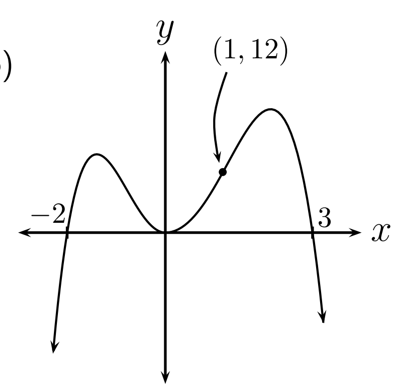
   (f) $f(x) = \frac{1}{2}x^2(x+3)(x-4)^2$

8. For each of the following, try to find the factored form of a polynomial function having the given graph. As usual, check yourself with your calculator.

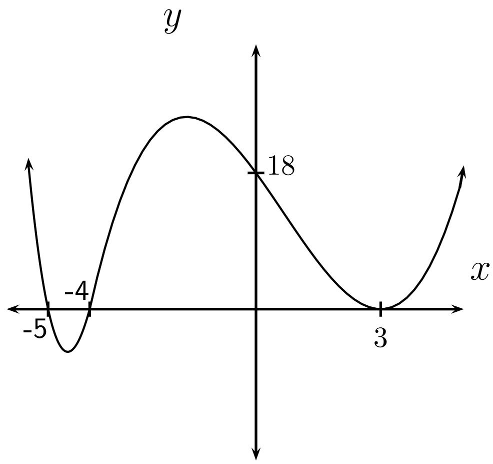
(a)



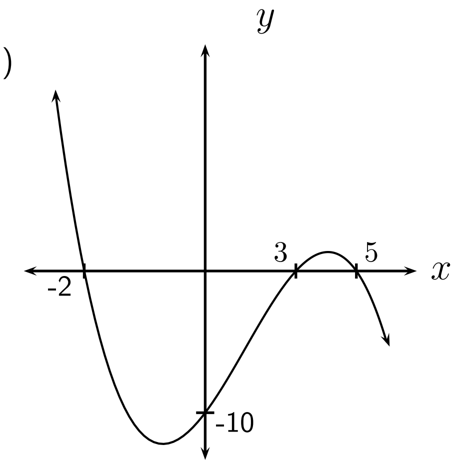
(b)



(c)



(d)



9. (a) - (d) Give the end behavior for each of the functions from Exercise 8.

4.3 Polynomial Inequalities

Performance Criteria:

4. (e) Solve a polynomial inequality.

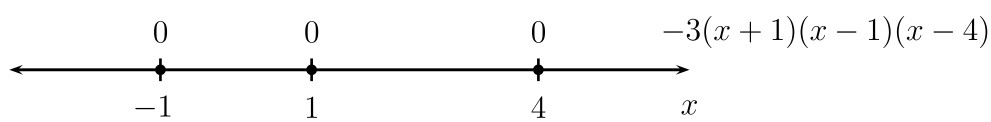
We already solved quadratic inequalities in Section 3.3. The method of solving inequalities containing polynomials of degree higher than 2 is essentially the same as the method we used for solving quadratic inequalities:

- Get zero alone on one side of the inequality and factor the nonzero side. (This will often already be done for you.)
- Determine where the polynomial is equal to zero and indicate those values on a number lines with open circles in the case of a strict inequality (¡ or ¿) or with solid dots in the case that the inequality allows equality as well ($\leq$ or $\geq$).
- Test a value in each of the intervals created by your points and shade that interval if the test value makes the inequality true.
- Looking at your number line, give the solution (all shaded points) using interval notation.

Let's take a look at an example:

◇ **Example 4.3(a):** Solve the inequality $-3(x+1)(x-1)(x-4) \leq 0$.

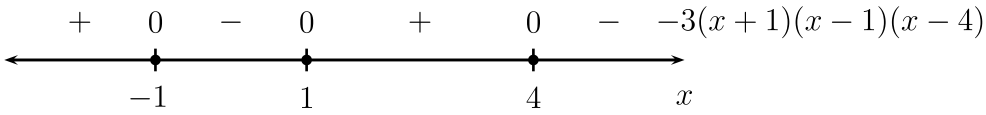
Solution: We begin by noting the values that make $-3(x+1)(x-1)(x-4) = 0$, which are $x = -1$, 1 and 4. Because the inequality includes "equal to," we indicate those values with solid dots on a number line:



The values below the number line are x values, and the numbers above indicate the value (or sign) of $-3(x+1)(x-1)(x-4)$. Next we need to determine the sign in each of the intervals between the indicated values using a table like that used in Example 4.2(c):

	-3	$(x+1)$	$(x-1)$	$(x-4)$		$-3(x+1)(x-1)(x-4)$
$x=-2:$	$(-)$	$(-)$	$(-)$	$(-)$	$=$	$+$
$x=0:$	$(-)$	$(+)$	$(-)$	$(-)$	$=$	$-$
$x=2:$	$(-)$	$(+)$	$(+)$	$(-)$	$=$	$+$
$x=5:$	$(-)$	$(+)$	$(+)$	$(+)$	$=$	$-$

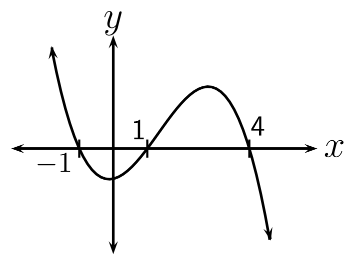
We can add the above signs to our number line, as shown at the top of the next page.



Because the inequality is true when $-3(x+1)(x-1)(x-4)$ is less than or equal to zero, we can see from the above number line that the solution to the inequality is $[-1,1] \cup [4,\infty)$.

Note that if we were to graph the function

$$y = -3(x+1)(x-1)(x-4)$$

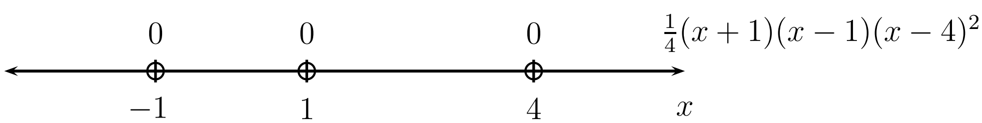


we would get the graph to the right. The solution to the inequality from the previous example is then all the intervals where the function is negative. One can see that those intervals are precisely the ones we arrived at from the sign chart.

Looking at the signs on our last number line from Example 4.3(a), we see that the sign changed every time we passed a point where the expression $-3(x+1)(x-1)(x-4)$ had the value zero. In the next example we will see that is not always the case. We will also use a slightly different and more efficient manner of showing the determination of the sign of the expression in each interval.

◇ **Example 4.3(b):** Solve the inequality $\frac{1}{4}(x+1)(x-1)(x-4)^2 > 0$.

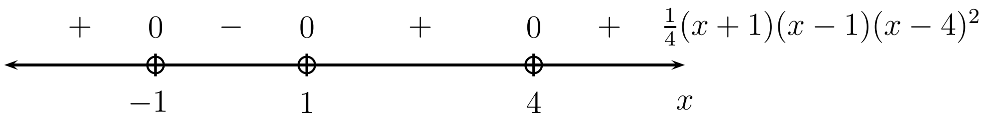
Solution: The the left side of this inequality is zero for the same values of x as the inequality in Example 4.3(a), but the fact that we have a strictly greater than inequality means that we need to use open circles at those points, giving us this initial number line:



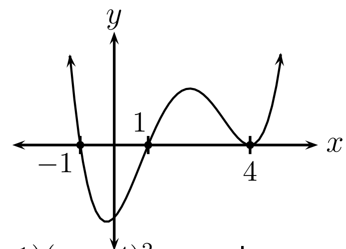
We'll test the same values as last time, but let's just make a simple table with the expression $\frac{1}{4}(x+1)(x-1)(x-4)^2$ at the top and the test values of x down the left side. In the table itself we'll give the sign of each factor of the expression for a given test value and multiply them to get the value of the entire expression, which will be at the right side of the table. This is all probably easier just seen than described, so here it is!

	$\frac{1}{4}$	$(x+1)$	$(x-1)$	$(x-4)^2$		$\frac{1}{4}(x+1)(x-1)(x-4)^2$
$x=-2$:	$(+)$	$(-)$	$(-)$	$(+)$	$=$	$+$
$x=0$:	$(+)$	$(+)$	$(-)$	$(+)$	$=$	$-$
$x=2$:	$(+)$	$(+)$	$(+)$	$(+)$	$=$	$+$
$x=5$:	$(+)$	$(+)$	$(+)$	$(+)$	$=$	$+$

Of course the factor $\frac{1}{4}$ is always positive, and the factor $(x-4)^2$ is always positive or zero (when $x=4$). The values to the right above give us the signs of the expression $\frac{1}{4}(x+1)(x-1)(x-4)^2$ for our test values, which we can record on our number line:



Keeping in mind that this is a strict inequality (NOT greater than or equal to), we are looking for only the values where $\frac{1}{4}(x+1)(x-1)(x-4)^2$ is positive, not zero. The solution is then the intervals $(-\infty,-1)$, $(1,4)$ and $(4,\infty)$. Note that $x=4$ itself is not included.



The graph of $y=\frac{1}{4}(x+1)(x-1)(x-4)^2$ is shown to the right. From it we can see that y is positive on precisely the intervals found above. How would our results change for the inequality

$$\frac{1}{4}(x+1)(x-1)(x-4)^2 \geq 0?$$

Well, now we include the values $x=-1,1,4,$ where $\frac{1}{4}(x+1)(x-1)(x-4)^2$ equals zero. This "plugs the hole" at $x=4$, giving the solution $(-\infty,-1]$ and $[1,\infty)$.

Section 4.3 Exercises

To Solutions

1. Solve the inequality $x(x-4)(x+1)\leq 0$. Don't forget to consider the factor x!

2. Solve each of the following polynomial inequalities. You will have to factor a couple of them yourself.

   (a) $(x+3)(x-1)(x-5)<0$

   (b) $(x+2)(x-2)(x-5)(x+3)\leq 0$

   (c) $0<(x+5)(x+1)^2$

   (d) $x^4-4x^2\geq 0$

   (e) $0\geq -x^3+10x^2-24x$

   (f) $(x+2)^2(x-3)^2>0$

4.4 Rational Functions, Part I

Performance Criteria:

4. (f) Give the x- and y-intercepts and the equations of the vertical and horizontal asymptotes of a rational function from either the equation or the graph of the function.

   (g) Graph a rational function from its equation, without using a calculator.

What IS a Rational Function?

Rational functions are functions that consist of one polynomial over another. Recognizing also that linear functions and constants qualify as polynomials, here are a few examples of rational functions:

$$f(x) = \frac{3x^2 - 14x - 5}{x^2 - 9}, \qquad g(x) = \frac{3x^2}{(x-2)^2}, \qquad y = -\frac{3}{x+5}.$$

In this section we will investigate the behaviors of rational functions through their graphs. Even though rational functions are quotients of two polynomial functions, *their graphs are significantly different than the graphs of polynomial functions!* Before beginning, let's make note of two facts:

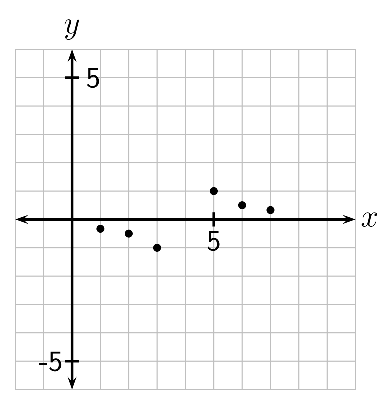
- A fraction is not allowed to have a denominator of zero.
- A fraction *CAN* have a numerator of zero, as long as the denominator is not zero at the same time; in that case the value of the entire fraction is zero. In fact, *the only way that a fraction can have value zero is if its numerator is zero.*

It is very important that you think about these two things and commit them to memory if you are to work comfortably with rational functions.

Graphs of Rational Functions, a First Look

Let's begin our investigation of rational functions with the rational function $h(x) = \frac{1}{x-4}$. The first observation we should make is that Dom$(f) = \{x \mid x \neq 4\}$. This means that above or below every point on the x-axis except four there will be a point on the graph of this function. The graph will then have two parts - a part to the left of where x is four and a part to the right of where x is four. Let's begin by finding a few function values for choices of x on either side of four and graphing them, as shown to the right. Note the inclusion of the value four for x and the indication in the table that $h(4)$ does not exist (DNE).

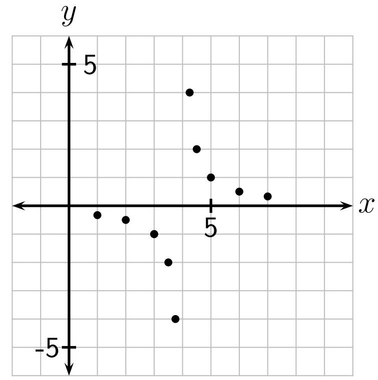
x	$h(x)$
1	$-\frac{1}{3}$
2	$-\frac{1}{2}$
3	-1
4	DNE
5	1
6	$\frac{1}{2}$
7	$\frac{1}{3}$



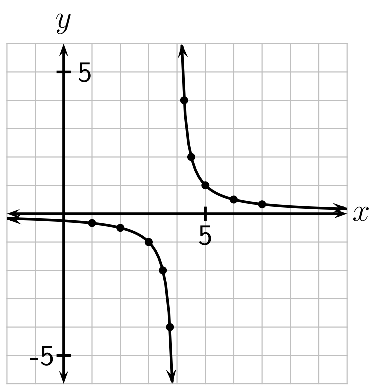
This picture is a bit baffling. There are two sets of points, each seeming to indicate a portion of a graph, but it is not clear what is happening between them. Lets consider some values of x on either side of four, getting "closer and closer" to four itself, working with decimals now. Such values of the function are shown in the two tables below and to the left, and below and to the right is our graph with some of the new points.

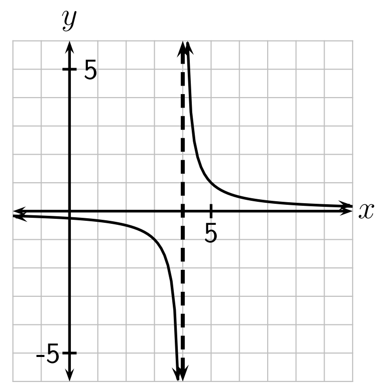
x	$h(x)$
3.5	-2
3.75	-4
3.9	-10
3.99	-100

x	$h(x)$
4.5	2
4.25	4
4.1	10
4.01	100



We can now see that our graph seems to consist of two parts, one on each side of where x has the value four. The graph is shown below and to the left; below and to the right we have added as a dashed line the graph of the vertical line with equation $x = 4$.

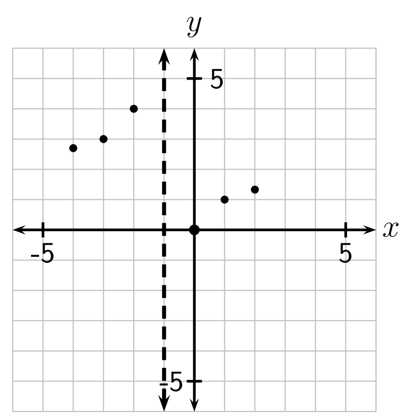




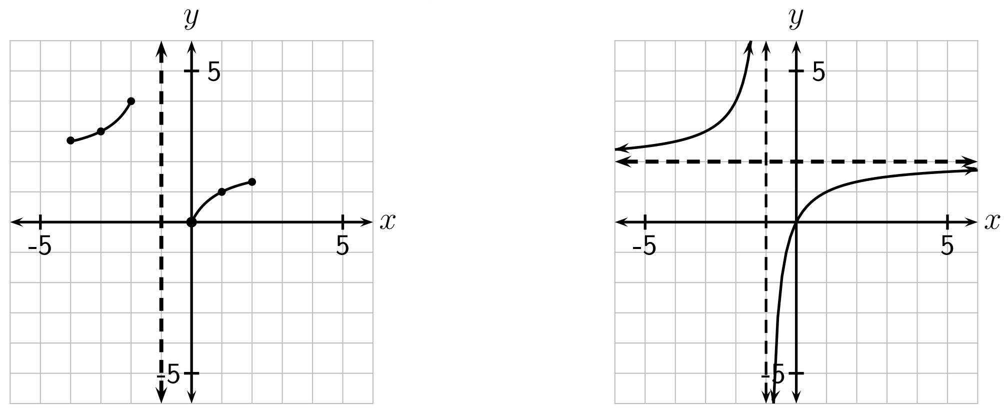
A line that a graph gets closer and closer to is called an **asymptote**. In this case the line $x = 4$ is what we will call a **vertical asymptote**. We can also see that as the graph goes off to the left or the right it gets closer and closer to the x-axis, which is the line $y = 0$. That line is of course a **horizontal asymptote**. It should be clear that the vertical asymptote is caused by the fact that x cannot have the value four, but the reason for the horizontal asymptote might not be as clear.

Let's look at another example to try to see how the equation might give us the horizontal asymptote; we'll use $g(x) = \dfrac{2x}{x+1}$. Because $\text{Dom}(g) = \{x \mid x \neq -1\}$ we expect a vertical asymptote with equation $x = -1$. To the right we plot the vertical asymptote, and a few points on either side of it.

x	$g(x)$
-4	2.7
-3	3
-2	4
-1	DNE
0	0
1	1
2	1.3



If we connect the points we have so far we get the graph shown below and to the left. It appears that the horizontal asymptote might be the line $y = 2$, and we will see that this is in fact the case. The final graph of $g(x) = \dfrac{2x}{x+1}$ is shown below and to the right.



Finding Asymptotes

By now it should be clear how to find the locations and equations of vertical asymptotes of a rational function; a vertical asymptote will occur at any value of x for which the denominator is zero. (This is a tiny lie - there is one unusual situation in which that is not the case, but for our purposes you can take it to be true.) To understand horizontal asymptotes let's begin by looking at the function $g(x) = \dfrac{2x}{x+1}$. Note that if the value of x is fairly large, $x + 1 \approx x$ and $g(x) \approx \dfrac{2x}{x} = 2$. (Here $\approx$ means *approximately equal to*.) The same thing occurs if x is a "large negative," meaning a negative number that is large in absolute value. Therefore, as we follow the ends of the graph toward negative and positive infinity for x, the values of y get closer and closer to two.

Consider the rational function $y = \dfrac{-3x^2 + 12x}{2x^2 - 2x - 12}$. In this case, the values of $12x$ and $-x$ are not small as x gets large (negative or positive), but they *ARE* small *relative to the values of* $-3x^2$ and $2x^2$. For example, if $x = 1000$, $12x = 12,000$ but $-3x^2 = -3,000,000$. The absolute value of $-3x^2$ is much larger than the absolute value of $12x$. (If you don't see this, just imagine the difference between having \$12,000 and three million dollars. \$12,000 wouldn't even buy one new automobile, whereas three million dollars would buy 150 cars that cost \$20,000 each. A similar disparity exists between $2x^2$ and $-2x$ in the denominator. Because of this, for the absolute value of x large (that is, if x is a large positive or negative number),

$$y = \frac{-3x^2 + 12x}{2x^2 - 2x - 12} \approx \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

The rational function $y = \dfrac{-3x^2 + 12x}{2x^2 - 2x - 12}$ then has a horizontal asymptote of $y = -\frac{3}{2}$. Note that the numerator value -3 came from the lead coefficient of the numerator, and the denominator value of 2 is the lead coefficient of the denominator.

So what about our first function, $h(x) = \dfrac{1}{x-4}$? In this case, for $|x|$ large we have $h(x) = \dfrac{1}{x-4} \approx \dfrac{1}{x} \approx 0$. This is why our graph had a horizontal asymptote of $y = 0$. Here is a summary of how we find the asymptotes of a rational function:

Finding Vertical and Horizontal Asymptotes

For the rational function $R = R(x)$,

- if the number a causes the denominator of R to be zero, then $x = a$ is a vertical asymptote of R. It is generally best to factor both the numerator and denominator of a rational function if possible; the values that make the denominator zero can then be easily seen.
- if the degree of the numerator of R is less than the degree of the denominator, R will have a horizontal asymptote of $y = 0$.
- if the degrees of the numerator and denominator of R are the same, then the horizontal asymptote of R will be $y = \frac{A}{B}$, where A and B are the lead coefficients of the numerator and denominator of R, respectively (including their signs!).

You might notice that we left out the case where the degree of the numerator is greater than the degree of the denominator. The appearance of the graph of such a rational function is a bit more complicated, and we won't address that issue.

◇ **Example 4.4(a):** Give the equations of the vertical and horizontal asymptotes of the rational function $y = \dfrac{3x-3}{x^2-2x-3} = \dfrac{3(x-1)}{(x+1)(x-3)}$. Horizontal Asymptote Example

Solution: We can see that if x was -1 or 3 the denominator of the function would be zero. Therefore the vertical asymptotes of the function are the vertical lines $x = -1$ and $x = 3$. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.

The following example shows something that a person needs to be a little cautious about.

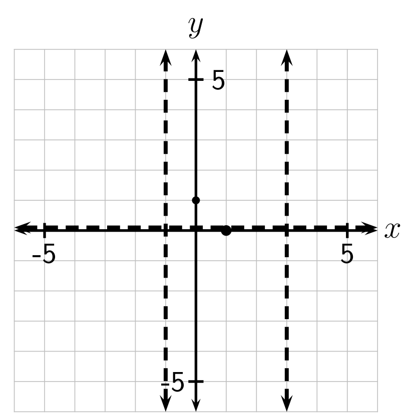
◇ **Example 4.4(b):** Give the equations of the vertical and horizontal asymptotes of the function $y = -\dfrac{x^2-4x+3}{x^2-4x+4} = -\dfrac{(x-1)(x-3)}{(x-2)^2}$.

Solution: Since only the value 2 for x results in a zero denominator, there is just one vertical asymptote, $x = 2$. The degree of the numerator is the same as the degree of the denominator, so the horizontal asymptote will be the *negative of* the lead coefficient of the numerator over the lead coefficient of the denominator, because of the negative sign in front of the fraction. Thus the horizontal asymptote is $y = -1$.

> **Graphing Rational Functions**
>
> To graph a rational function,
>
> 1) Find all values of x that cause the denominator to be zero, and graph the corresponding vertical asymptotes.
> 2) Find and graph the horizontal asymptote.
> 3) Find and plot the x-intercepts, keeping in mind that a fraction can be zero only if its *numerator* is zero.
> 4) Find and plot the y-intercept.
> 5) Sketch in each part of the graph, either by using reasoning and experience with what such graphs look like, or by plotting a few more points in each part of the domain of the function.

◇ **Example 4.4(c):** Sketch the graph of $y = \dfrac{3x-3}{x^2-2x-3} = \dfrac{3(x-1)}{(x+1)(x-3)}$, indicating all asymptotes and intercepts clearly and accurately. Another Example

Solution: We can see that if x was -1 or 3 the denominator of the function would be zero. Therefore the vertical asymptotes of the function are $x=-1$ and $x=3$. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y=0$.

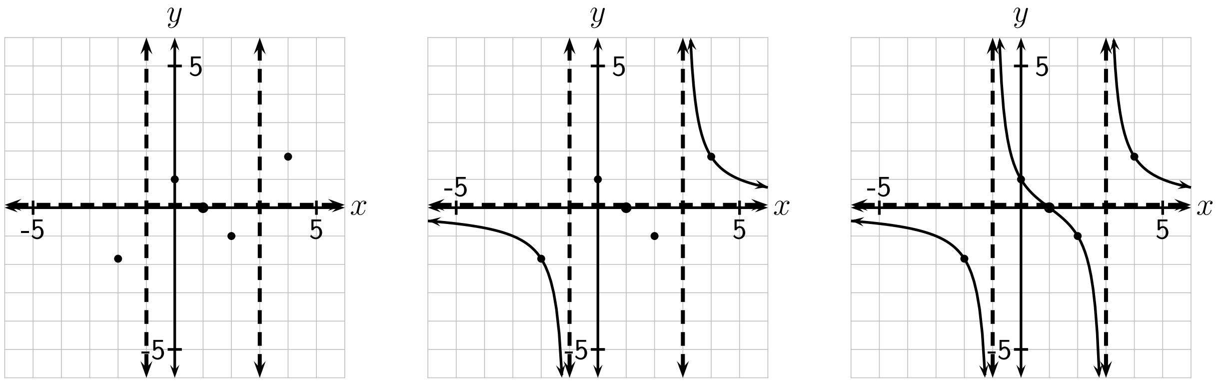


Next we find the intercepts. The only way that y can be zero is if $x=1$, So we have an x-intercept of $(1,0)$. If $x=0$ then $y=1$, so our y-intercept is $(0,1)$. We now begin constructing our graph by sketching in the asymptotes and plotting these two intercepts, as shown to the right.

To fill in the rest of the graph we might wish first to plot a few more points. Good locations to have more information would be "outside" the two vertical asymptotes, and between the x-intercept and the vertical asymptote $x=3$. Computing y for $x=-2$, 2 and 4 gives us the points $(-2,-\frac{9}{5})$, $(2,-1)$ and $(4,\frac{9}{5})$. The graph to the left at the top of the next page shows the above graph with those points added.

Now consider the point $(4,\frac{9}{5})$. As the graph goes leftward from that point it will be "pushed" either up or down by the vertical asymptote at $x=3$. Because there is no

x-intercept between $x = 3$ and $x = 4$, the graph must go upward as it moves left from $(4, \frac{9}{5})$. As it moves right from that point it must approach the horizontal asymptote of $y = 0$, But the graph won't cross $y = 0$ because there are no x-intercepts to the right of $x = 4$. Similar reasoning can be used to the left of the vertical asymptote $x = -1$; from these things we get the graph shown in the middle below.

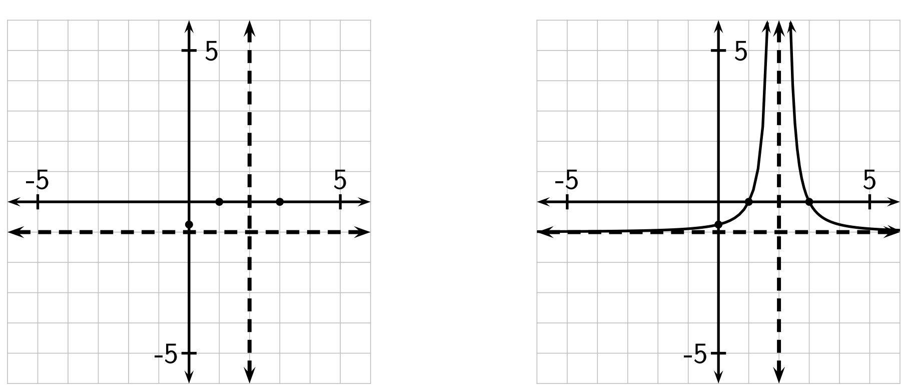


We now need only to complete the middle portion of the graph, between the two asymptotes. The three points are on a line, and to the left of the y-intercept the graph must bend upward because if it bent downward there would have to be another x-intercept between $x = -1$ and $x = 0$. By the same reasoning the graph must bend downward from $(2, -1)$, so the final graph has the appearance shown above and to the right.

The above example illustrates something important: that the graph of a rational function *CAN* cross a *horizontal* asymptote, and it often does.

◇ **Example 4.4(d):** Sketch the graph of $y = -\dfrac{x^2 - 4x + 3}{x^2 - 4x + 4} = -\dfrac{(x-1)(x-3)}{(x-2)^2}$, indicating all asymptotes and intercepts clearly and accurately.

Solution: In this case there is just one vertical asymptote, $x = 2$, and the horizontal asymptote is $y = -1$. (See Example 4.4(b).) $y = 0$ when $x = 1$ or 3, so the x-intercepts are 1 and 3. The y-intercept is $-\frac{3}{4}$. Plotting all of this information gives the graph shown below and to the left. Using the same kinds of reasoning as in the previous example we can then complete the graph as shown below and to the right.



Section 4.4 Exercises

To Solutions

1. For each of the rational functions below,

   (i) give all x-intercepts

   (ii) give all y-intercepts

   (iii) give the *equations* of all vertical asymptotes

   (iv) give the *equation* of the horizontal asymptote

   Note that some are given in both standard and factored forms, each of which might be more useful for one or the other of the above.

   (a) $f(x) = \dfrac{6}{x-3}$

   (b) $y = \dfrac{2x}{x+1}$

   (c) $g(x) = -\dfrac{3}{x+5}$

   (d) $h(x) = \dfrac{-2x+6}{x+1}$

   (e) $y = \dfrac{x^2-2x-3}{x^2+2x-8} = \dfrac{(x+1)(x-3)}{(x+4)(x-2)}$

   (f) $f(x) = \dfrac{2x^2-4x-30}{x^2-x-2} = \dfrac{2(x+3)(x-5)}{(x+1)(x-2)}$

   (g) $g(x) = \dfrac{x^2+3x-4}{x^2-4} = \dfrac{(x+4)(x-1)}{(x+2)(x-2)}$

   (h) $h(x) = \dfrac{x+2}{x^2-2x-3} = \dfrac{x+2}{(x+1)(x-3)}$

   (i) $g(x) = \dfrac{4}{x^2-6x+9} = \dfrac{4}{(x-3)^2}$

   (j) $h(x) = \dfrac{2x^2+7x+5}{x^2+3x-4} = \dfrac{(2x+5)(x+1)}{(x+4)(x-1)}$

   (k) $h(x) = \dfrac{-x^2+4}{x^2-3x-4} = \dfrac{-(x+2)(x-2)}{(x+1)(x-4)}$

   (l) $h(x) = \dfrac{x+1}{x^2-5x} = \dfrac{x+1}{x(x-5)}$

2. Graph each of the rational functions from Exercise 1. Check your answers using technology.

3. In each part of this exercise you will be asked to create the equation of a rational function whose graph has certain characteristics. Each new function can be created by just making some modification of the previous function. At each step, check your graph with a graphing program to make sure it has the correct characteristics. It might help to zoom out some when checking the asymptotes.

   (a) Create the equation of a rational function $f(x)$ that has a single vertical asymptote of $x = 2$.

(b) Create the equation of a rational function that has a vertical asymptote of $x = 2$ and an x-intercept of $x = 4$.

(c) Create the equation of a rational function that has a vertical asymptote of $x = 2$, an x-intercept at $x = 4$, and a horizontal asymptote of $y = -3$.

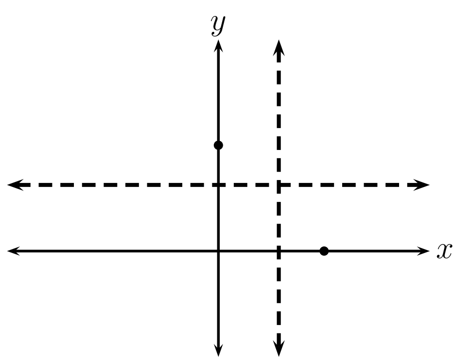
4. Create the equation of a rational function that has

- vertical asymptotes of $x = -1$ and $x = 3$
- a horizontal asymptote of $y = 2$
- x-intercepts of $x = 0$ and $x = 4$

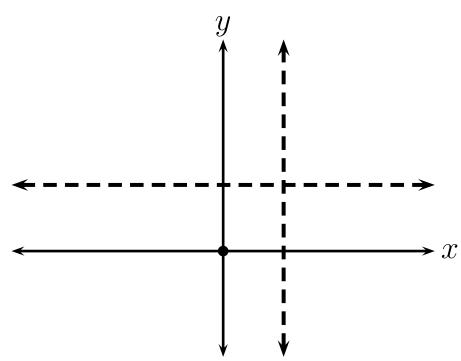
Check your answer with a graphing utility.

5. Each of the graphs below shows *the only* intercepts and asymptotes for a rational function. Using them, sketch what a graph of the function would probably look like. If there is no horizontal asymptote shown, assume that it is $y = 0$.

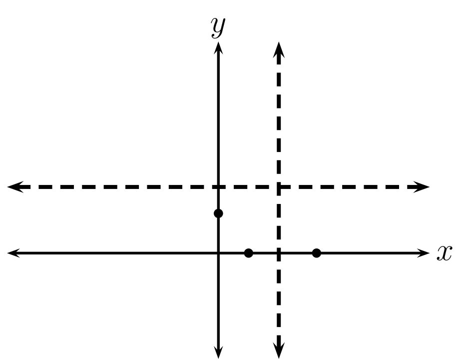
(a)



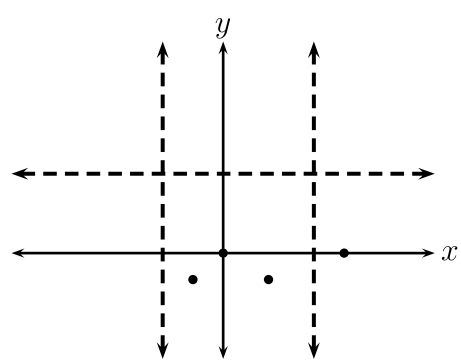
(b)



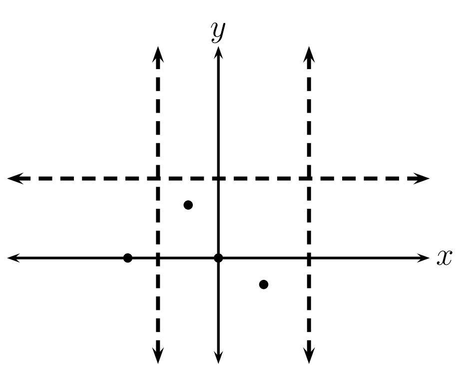
(c)



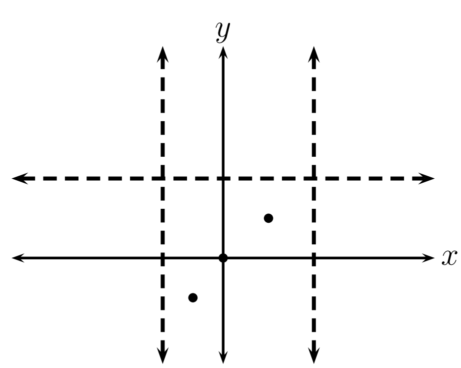
(d)



(e)



(f)



6. Recall that the cost (in dollars) for the Acme Company to produce x Widgets in a week was given by the equation $C = 7x + 5000$.

   (a) What is the cost of producing 5,000 Widgets in one week? What is the *average* cost per widget when 5,000 Widgets are produced in a week?

   (b) What is the average cost per Widget if $10,000$ are produced in one week?

   (c) Explain any difference in your answers to (a) and (b).

   (d) Write an equation that gives the average cost $\bar{C}$ per widget produced in one week, again using x to represent the number of Widgets produced in that week.

   (e) Your answer to (d) is a rational function. Discuss its domain (both the mathematical domain and the realistic domain).

   (f) Give any vertical asymptotes of the function and describe their meaning(s) in terms of costs and Widgets.

   (g) Give any horizontal asymptotes of the function and describe their meaning(s) in terms of costs and Widgets.

4.5 Rational Functions, Part II

Performance Criteria:

4. (h) Give the end behavior and behavior of a rational function from its graph, using "as $x \to a$, $f(x) \to b$ notation, where a is a real number, $-\infty$ or ∞.

When working with a polynomial function like $y = x^3 + 5x^2 - 7$, we acknowledge that the domain of the function is all real numbers. A consequence of this is that if we want to know something about the value of the function at a value of x we can simply find the y value that corresponds to that x value, regardless of what it is. On the other hand, consider the function $f(x) = \dfrac{1}{(x-3)^2}$, whose graph is shown below and to the left. It should be clear that the domain of the function is all real numbers except for three; because of this, the value of $f(3)$ cannot be found. In spite of this, we would like to be able to describe the behavior of the function when x gets "near" three. To do this, we need to recall that there are "places" at the left and right ends of the x-axis that we call negative infinity and infinity (denoted by $-\infty$ and ∞), and similarly for the ends of the y-axis. These "places" are "beyond all numbers," no matter how large or small. You saw this when we were using interval notation to define domains and ranges of functions and, more recently, when we described end behavior of polynomial functions in Section 4.1.

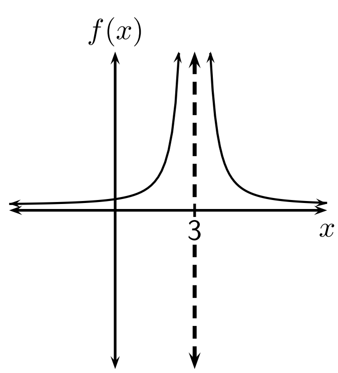
Given this idea, we are now ready to describe the behavior of the function f near $x = 3$. What we say is that "as x approaches three, $f(x)$ approaches infinity." Symbolically we will write the same thing this way:

$$\text{As } x \to 3,\ f(x) \to \infty$$

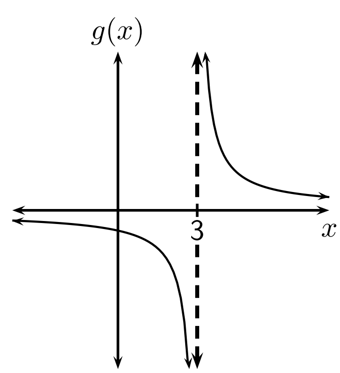
We also write similar statements to describe what happens as the graph leaves the picture to the left and right:

$$\text{As } x \to -\infty,\ f(x) \to 0 \qquad \text{and} \qquad \text{As } x \to \infty,\ f(x) \to 0$$

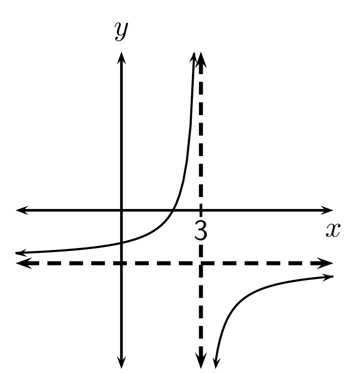
Since the function does the same thing regardless of which infinity x approaches, these two statements can be combined into one: As $x \to \pm\infty$, $f(x) \to 0$.



$f(x) = \dfrac{1}{(x-3)^2}$



$g(x) = \dfrac{1}{x-3}$



$y = \dfrac{2x}{3-x}$

Now look at the graph of $g(x) = \dfrac{1}{x-3}$ on the previous page. Hopefully you can see that there is a problem with the statement "As $x \to 3$, $f(x) \to \infty$," since the function goes to either ∞ or $-\infty$, depending on which side we come at three from. When x approaches three and is larger than three, we say that it approaches three "from above," or "from the positive side." This is denoted boy $x \to 3^+$. Similarly, when x approaches three from the left, or from below, we write $x \to 3^-$. Using these notations, we can now solve our dilemma by writing

$$\text{As } x \to 3^-,\ g(x) \to -\infty \qquad \text{and} \qquad \text{As } x \to 3^+,\ g(x) \to \infty$$

Also, as $x \to \pm\infty$, $g(x) \to 0$.

Finally, look at the graph of $y = \dfrac{2x}{3-x}$ on the previous page and try to write statement like we have been, for it. You should come up with

$$\text{As } x \to 3^-,\ y \to \infty, \qquad \text{as } x \to 3^+,\ y \to -\infty \qquad \text{and} \qquad \text{as } x \to \pm\infty,\ y \to -2.$$

You should think about how these three examples compare with each other. Note in particular that when the function does different things as x approaches a value from different directions we must write two separate statements indicating this. But when the function does the same thing regardless of which side x approaches a value from we can write just one statement, like the statement "as $x \to 3$, $f(x) \to \infty$" we wrote for the function f on the previous page.

Section 4.5 Exercises

To Solutions

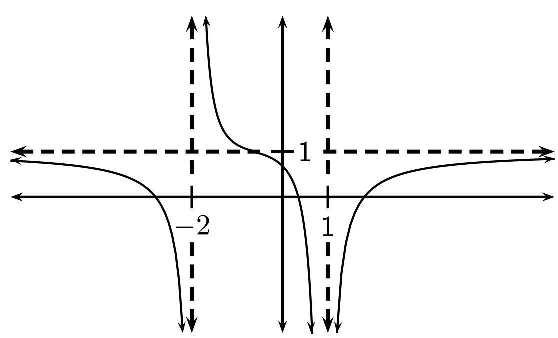
1. Fill in the blanks for the function whose graph is shown below and to the right.

   As $x \to -2^-$, $y \to$ _____

   As $x \to -2^+$, $y \to$ _____

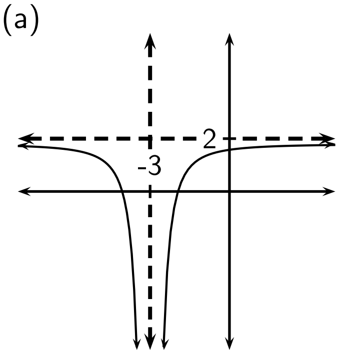
   As $x \to 1$, $y \to$ _____

   As $x \to \pm\infty$, $y \to$ _____

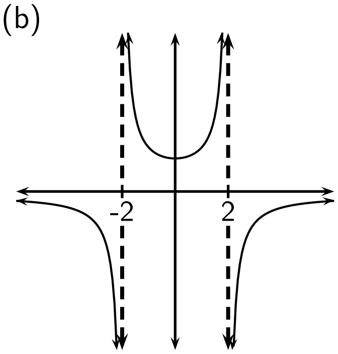


2. Statements of the form "as $x \to a$, $y \to b$" are called **limit statements**. For each of the functions whose graphs are shown, write all appropriate limit statements.

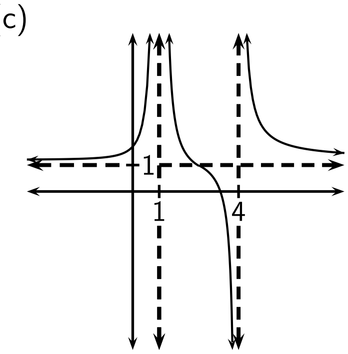
(a)



(b)



(c)



3. The domain of a rational function is all real numbers except any where vertical asymptotes occur. You should have seen this in Exercises 1 and 2 of the previous section. So if we look at the graph for Exercise 2(a) we can see that the domain of the graphed function is $\{x \mid x \neq -3\}$. We can determine the range of such a function from its graph by the method described near the end of Section 2.2. The range is basically all the sets on the y-axis that are horizontally aligned with some point on the graph. The range of the function graphed in Exercise 2(a) is $(-\infty, 2)$ or $\{y \mid y < 2\}$. Graph each of the following functions using some technology and use the graph of each to determine its range.

   (a) $y = \dfrac{1}{x-2}$
   (b) $f(x) = \dfrac{1}{(x-2)^2}$
   (c) $g(x) = \dfrac{x}{2-x}$
   (d) $h(x) = \dfrac{x^2}{(x-2)^2}$
   (e) $y = \dfrac{x^2}{x^2-4}$
   (f) $g(x) = \dfrac{x}{x^2-4}$

4. Give the interval(s) on which each of the functions from Exercise 3 is increasing.

5. Give the interval(s) on which each of the functions from Exercise 3 is positive.

6. Determine whether each function from Exercise 3 has any minima or maxima. For any that do, give each minimum/maximum, its location, and whether it is relative or absolute.

5 More on Functions

Outcome/Performance Criteria:

5. Create new functions from existing functions, understand and find inverse functions.the effect of certain compositions on the graphs of functions. Understand the effect of certain compositions on the graphs of functions.

   (a) Find the sum, difference, product or quotient of two functions.

   (b) Determine the domain of a sum, difference, product or quotient of two functions.

   (c) Find the partial fraction decomposition of a rational function.

   (d) Compute and simplify the composition of two functions.

   (e) Determine functions whose composition is a given function.

   (f) Use composition to determine whether two functions are inverses of each other.

   (g) Compute the inverse of a function.

   (h) Given the graph of a function, draw or identify the graph of its inverse.

   (i) Determine whether a function is one-to-one based on its graph.

   (j) When necessary, restrict the domain of a function so that it has an inverse. Give the inverse and its domain.

   (k) Identify the graphs of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

   (l) Given the graph of a function, sketch or identify various transformations of the function.

5.1 Combinations of Functions

Performance Criteria:

5. (a) Find the sum, difference, product or quotient of two functions.
   (b) Determine the domain of a sum, difference, product or quotient of two functions.
   (c) Find the partial fraction decomposition of a rational function.

Given two functions f and g we can use them to create a new function that we'll call $f+g$. The first thing we need when working with a new function is some sort of description of how it works, often given as a formula. So, for any number x, we need to know how to find $(f+g)(x)$. The idea is simple: we find $f(x)$ and $g(x)$, and add the two together. That is, $(f+g)(x) = f(x)+g(x)$. If we have equations for f and g, we will usually combine them to get an equation for $f+g$, as shown in the next example.

◇ **Example 5.1(a):** For $f(x) = x-1$ and $g(x) = 3x^2+2x-5$, find the equation for $(f+g)(x)$.

Solution: $(f+g)(x) = f(x) - g(x) = (x-1)+(3x^2+2x-5) = 3x^2+3x-6$

Note that we are simply finding an equation for a new function, so the result should just be an expression with the unknown in it. There is no need to factor it unless asked to, and *we should definitely NOT set it equal to zero* and solve. It should be clear that we can define functions $f-g$, fg and $\frac{f}{g}$ in a similar manner. Their definitions are

$$(f-g)(x) = f(x)-g(x), \qquad (fg)(x) = f(x)g(x), \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Technically there are two more functions here, $g-f$ and $\frac{g}{f}$, since subtraction and division are not commutative. That is, changing the order of subtraction or division changes the result.

◇ **Example 5.1(b):** Again let $f(x) = x-1$ and $g(x) = 3x^2+2x-5$. Find the equations for $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$. Multiplying and Dividing Examples

Solution: $(f-g)(x) = (x-1)-(3x^2+2x-5) = -3x^2-x+4$

$$(fg)(x) = (x-1)(3x^2+2x-5) = 3x^3+2x^2-5x-3x^2-2x+5 = 3x^3-x^2-7x+5$$

$$\left(\frac{f}{g}\right)(x) = \frac{x-1}{3x^2+2x-5} = \frac{x-1}{(3x+5)(x-1)} = \frac{1}{3x+5}$$

Note that some care needs to be taken to distribute minus signs when computing $f-g$ or $g-f$.

Domains of Combinations of Functions

Suppose that we wish to construct one of the functions $f+g$, $f-g$, $g-f$, or fg from two functions f and g. It should be clear that such a function shouldn't be valid for values of x for which one or the other (or both) of f and g are not valid. Another way of saying this is to say that a value x is in the domain of one of those functions as long as it is the domain of *BOTH* f and g. The domain of f is a set of numbers, and the domain of g is another set of numbers. The domain of $f+g$, $f-g$, $g-f$, or fg is then the set of numbers that contains all numbers in the domains of both f and g. Such a new set constructed out of two known sets has a name:

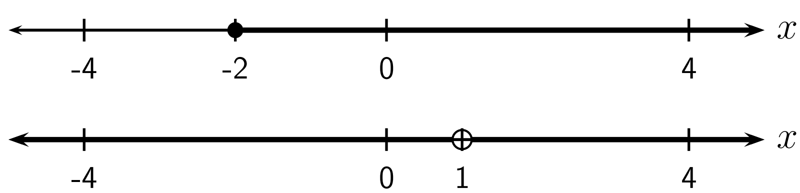
> **Intersection of Two Sets**
>
> For two sets A and B, the **intersection** of A and B is the set containing all numbers that are in *both* A and B. We denote this new set by $A \cap B$.

Using this new concept and its notation, along with the notation $\text{Dom}(f)$ for the domain of a function f, we have

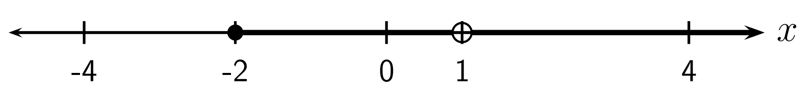
$$\text{Dom}(f+g) = \text{Dom}(f-g) = \text{Dom}(g-f) = \text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g).$$

◇ **Example 5.1(c):** Let $f(x) = \sqrt{x+2}$ and $g(x) = \dfrac{1}{x-1}$. Give the domain of $(g-f)(x)$.

Solution: We can see that $\text{Dom}(f) = [-2, \infty)$ and $\text{Dom}(g) = \{x \,|\, x \neq 1\}$. We can easily visualize the intersection of those two sets if we show them each on a number line. The top number line shows the domain of f, and the one below it shows the domain of g.



The domain of $g-f$ is then the intersection of those two sets, which means the points that are shaded on *both* number lines:



We can now give the domain of $g-f$, using interval notation, as $[-2, 1)$ together with $(1, \infty)$.

Creating the intersection of two sets is an operation on the two sets that yields a new set, in the same way that addition or multiplication are operations on two numbers that yield a new number. We now introduce another operation for combining two sets in a different way:

Union of Two Sets

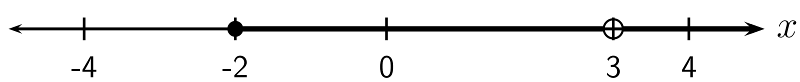
For two sets A and B, the **union** of A and B is the set containing all numbers that are in *either* A or B (including those points that are in both). We denote this new set by $A \cup B$.

Using this idea and notation, the domain of $g - f$ from the Example 5.1(c) is $[-2, 1) \cup (1, \infty)$. Note that we are not saying that the domain of $g - f$ is the union of the domains of f and g! We obtain the domain of $g - f$ by taking the intersection of the domains of the two functions f and g, but the result of that intersection is most cleanly described as the union of two sets (that are *not necessarily the domains of f and g*).

The domain of $\frac{f}{g}$ (and, of course, $\frac{g}{f}$), is "almost" the intersections of the domains of f and g, but there is an additional consideration. Any value in both the domain of f and the domain of g will be in the domain of $\frac{f}{g}$, *as long as it doesn't make the value of g zero*. So we find the domain of $\frac{f}{g}$ by taking the intersection of the domains of f and g and removing any values of x for which $g(x) = 0$.

◇ **Example 5.1(d):** Let $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 9$. Give the domain of $(\frac{f}{g})(x)$.

Solution: We know that $\mathrm{Dom}(f) = [-2, \infty)$ and $\mathrm{Dom}(g)$ is all real numbers. The intersection of these two sets is just $[-2, \infty)$. However, we see that $g(x) = x^2 - 9 = (x+3)(x-3)$ is zero when $x = -3$ and when $x = 3$. Therefore we need to remove the point $x = 3$ from the interval $[-2, \infty)$, giving us a set that looks like this:



So $\mathrm{Dom}(\frac{f}{g}) = [-2, 3) \cup (3, \infty)$.

Partial Fraction Decomposition

There are times when we wish to take a function of the form $f(x) = \dfrac{Ax + B}{(x - x_1)(x - x_2)}$ and find two functions of the form

$$g(x) = \frac{C}{x - x_1} \qquad \text{and} \qquad h(x) = \frac{D}{x - x_2}$$

such that f is the sum of g and h. We illustrate the method for doing this in the next example.

◇ **Example 5.1(e):** Let $f(x) = \dfrac{x+17}{(x-1)(x+5)}$. Find two functions g and h of the form given above such that $f(x) = (g+h)(x)$. Another Example

Solution: First we let $g(x) = \dfrac{C}{x-1}$ and $h(x) = \dfrac{D}{x+5}$, and add them together:

$$g(x)+h(x) = \frac{C}{x-1} + \frac{D}{x+5} = \frac{C(x+5)}{(x-1)(x+5)} + \frac{D(x-1)}{(x-1)(x+5)} = \frac{Cx+5C+Dx-D}{(x-1)(x+5)}$$

Now if $f(x)$ is to equal $g(x)+h(x)$, the last fraction on the line above must equal $\dfrac{x+17}{(x-1)(x+5)}$. Note that both fractions have the same denominator, so the two fractions will be equal only if their numerators are equal:

$$Cx + 5C + Dx - D = x + 17$$

By "grouping like terms," this can be rewritten (be sure you see how) as

$$(C+D)x + (5C-D) = 1x + 17,$$

and these will be equal only if $C + D = 1$ and $5C - D = 17$. Now we have two equations in two unknowns, which we know how to solve (see Section 1.6). If we add the two equations together we get $6C = 18$, so $C = 3$. Substituting this into the first equations gives $D = -2$. Thus

$$g(x) = \frac{3}{x-1} \qquad \text{and} \qquad h(x) = \frac{-2}{x+5}.$$

It is verified in the exercises that $f(x) = g(x) + h(x)$.

The method of partial fractions has many complications that can arise when the rational function f has other forms. Those of you who encounter partial fraction decomposition again will learn at that time how to alter the above technique for those forms.

Section 5.1 Exercises

To Solutions

1. Consider the functions $g(x) = 3x - 1$ and $h(x) = 2x + 5$.

   (a) Find and simplify $(g+h)(x)$.

   (b) Find and simplify $(gh)(x)$.

   (c) Find and simplify $\left(\dfrac{h}{g}\right)(x)$.

   (d) Give the domain of each of the above functions from parts (a), (b) and (c).

2. Let $f(x) = x^2 + 2x - 3$ and $g(x) = x^2 - 1$.

   (a) Find and simplify $(g - f)(x)$.

   (b) Find and simplify $(fg)(x)$.

   (c) Find and simplify $\left(\frac{f}{g}\right)(x)$.

   (d) Give the domains of each of the functions from parts (a), (b) and (c).

3. For $f(x) = \dfrac{x+5}{x-2}$ and $g(x) = \dfrac{x-4}{x-2}$, find and simplify $(f - g)(x)$. What is the domain of the function?

4. Let $f(x) = \dfrac{2x}{x-4}$ and $g(x) = \dfrac{x}{x+5}$. Find and simplify $\left(\frac{f}{g}\right)(x)$, and give its domain.

5. Let $f(x) = \dfrac{x+3}{x^2-25}$ and $g(x) = \dfrac{7}{x-5}$.

   (a) Find and simplify $(f + g)(x)$. What is the domain of the function?

   (b) Find and simplify $\left(\frac{g}{f}\right)(x)$. What is the domain of the function?

6. Letting $g(x) = \dfrac{3}{x-1}$ and $h(x) = \dfrac{-2}{x+5}$, find $(g + h)(x)$. Check your answer with the $f(x)$ from Example 5.1(e).

7. For each function f that is given, find functions g and h for which $f(x) = g(x)+h(x)$. That is, find the partial fraction decomposition of f.

   (a) $f(x) = \dfrac{4x+7}{x^2+5x+6}$

   (b) $f(x) = \dfrac{-14}{x^2-3x-10}$

   (c) $f(x) = \dfrac{11-x}{x^2-x-2}$

   (d) $f(x) = \dfrac{4x-10}{x^2-1}$

5.2 Compositions of Functions

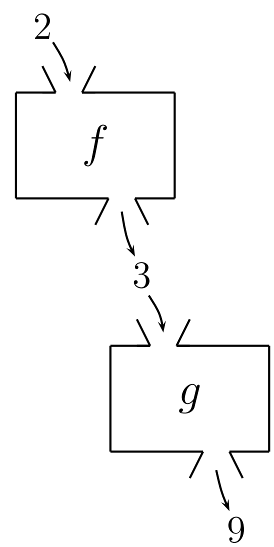
Performance Criteria:

5. (d) Compute and simplify the composition of two functions.
   (e) Determine functions whose composition is a given function.

There is another way of combining two functions, called their **composition**. Compositions of functions are far more important than their sums, differences, products and quotients, for various reasons. Let's introduce the concept using the two functions $f(x) = 2x - 1$ and $g(x) = x^2$. Suppose that we are asked to find $g[f(2)]$; what this means is that we are to first find $f(2)$, then we are to find g of that result. We can see that $f(2) = 2(2) - 1 = 3$, and $g(3) = 3^2 = 9$, so $g[f(2)] = 9$. We can do all of this as one computation, as follows:

$$g[f(2)] = g[2(2) - 1] = g[3] = 3^2 = 9$$

Note that the function that is closest to 2 *in the expression* $g[f(2)]$ *is the first to act!* The picture to the right shows how this can be interpreted in terms of the "machine" idea of a function. The number two is first "fed into" machine f, which gives an output of three. That value is then "fed into" g, which outputs nine, the final result.



- **Example 5.2(a):** For the same two functions $f(x) = 2x - 1$ and $g(x) = x^2$, find $f[g(2)]$. Is $f[g(2)] = g[f(2)]$?

$$f[g(2)] = f[2^2] = f(4) = 2(4) - 1 = 7$$

Solution: Clearly $f[g(2)]$ is not equal to $g[f(2)]$.

This exercise illustrates an important fact:

> *Suppose we have two functions, and we apply one of them to a number and the other one to the result. Changing the order in which the two functions operates generally changes the final result.*

Consider again Example 5.2(a), where we found $f[g(2)] = 7$. We did this by first applying g to the number two, giving the result of four. We then applied f to that value to obtain the value seven. We will now create a new function that does all of that in one step, and we'll call it the **composition of** f **with** g, denoted by $f \circ g$. We get it by simply letting f act on $g(x)$:

$$(f \circ g)(x) = f[g(x)] = f(x^2) = 2(x^2) - 1 = 2x^2 - 1.$$

Note then that

$$(f \circ g)(2) = 2(2)^2 - 1 = 2(4) - 1 = 7,$$

the result that we obtained in Example 5.2(a).

◇ **Example 5.2(b):** For the same two functions $f(x) = 2x - 1$ and $g(x) = x^2$, find $(g \circ f)(x)$. Then find $(g \circ f)(2)$.

Solution: $(g \circ f)(x) = g[f(x)] = g(2x - 1) =$

$$(2x - 1)^2 = (2x - 1)(2x - 1) = 4x^2 - 4x + 1$$

and

$$(g \circ f)(2) = 4(2)^2 - 4(2) + 1 = 4(4) - 8 + 1 = 9.$$

Before going on, let's conduct a little "thought exercise." Suppose that we made the claim that "all cows are brown." If we then went out and found one cow, and it was brown, would that prove our point? Of course not, but at least it would not disprove it either. If the first (or any) cow that we found when we went looking was black, one would have to conclude that our assertion was false.

Similarly, we can note that the value of $(g \circ f)(2)$ found in this last example is the same thing we got at the top of the previous page when we found $g[f(2)]$, except that in Example 5.2(d) we did it all with one function, rather than applying f, then g. *This does not show that the function $f \circ g$ we found is in fact correct, but at least it shows it is not incorrect either!* In other words, our function $f \circ g$ worked as it should on at least one number, so there is a reasonable chance that it is correct. This gives us a way of partially checking our answers when we create a composition.

NOTE: We sometimes speak of finding the composition of functions f and g as *composing f and g*. Some people might take this phrase to mean finding $f \circ g$, but when we use it there will be no order implied. That is, composing f and g will be taken to mean finding either $f \circ g$ or $g \circ f$, depending on the context or what is asked for.

In Chapter 3 we worked with functions like $y = 2(x - 4)^2 + 1$. If we were to be given a value for x and asked to find y, we could think of it as a three step process:

(1) subtract four from x

(2) square the result of (1)

(3) take the result from (2), multiply it by two and add one

Each of the above steps is a function in its own right:

$$(1)\ \ h(x) = x - 4 \qquad\qquad (2)\ \ g(x) = x^2 \qquad\qquad (3)\ \ f(x) = 2x + 1$$

Based on these, we see that

$$(f \circ g \circ h)(x) = f\{g[h(x)]\} = f\{g[x - 4]\} = f\{(x - 4)^2\} = 2(x - 4)^2 + 1 = y$$

We could actually break step (3) of multiplying by two and adding one into two separate steps, to get y as a composition of four functions.

◇ **Example 5.2(c):** Let $g(x) = \sqrt{x}$. Find two functions f and h such that the function $y = \dfrac{\sqrt{3x+5}}{2}$ is the composition $(f \circ g \circ h)(x)$

Solution: Here we need to think about the journey that x takes when we have a value for it and we are computing y. Before using the square root, which is function g, we need to multiply x by three and add five. Since h is the first function in $f \circ g \circ h$ to act on x, we have $h(x) = 3x+5$. Letting g act on that result gives us that $(g \circ h)(x) = \sqrt{3x+5}$. To get y we still need to divide by two, and that should be function f, so $f(x) = \frac{x}{2}$.

Section 5.2 Exercises

To Solutions

1. Consider the functions $g(x) = 3x - 1$ and $h(x) = 2x + 5$.

   (a) Find $g[h(-4)]$. Then find $(g \circ h)(x)$ and use that answer to find $(g \circ h)(-4)$. If you don't get the same thing that you did when you computed $g[h(-4)]$, go back and check your work for finding $(g \circ h)(x)$.

   (b) Find $h[g(7)]$, then find $(h \circ g)(x)$. Use the value $x = 7$ to check your answer, in the same way that you did in part (d).

2. $f(x) = x^2 - 5x$ and $g(x) = x - 1$. Find and simplify $(f \circ g)(x)$ and $(g \circ f)(x)$.

3. Let $f(x) = x - 2$, $g(x) = |x|$ and $h(x) = 3x + 1$. Find $(h \circ g \circ f)(x)$ and $(f \circ g \circ h)(x)$. **There is no simplifying that can be done here, so don't do any!**

4. For all parts of this exercise, let $f(x) = \sqrt{x}$.

   (a) Find a function g that can be composed with f to get the function $y = \sqrt{x+7}$. In this case, is $y = (f \circ g)(x)$, or is $y = (g \circ f)(x)$?

   (b) Find a function h that can be composed with f to get the function $y = 3\sqrt{x} - 2$. In this case, is $y = (f \circ h)(x)$, or is $y = (h \circ f)(x)$?

   (c) Find two *NEW* functions g and h such that $(g \circ f \circ h)(x) = \frac{1}{3}\sqrt{x-2} + 3$.

5. Give three functions f, g and h such that $(f \circ g \circ h)(x) = \frac{1}{2}(x-2)^3 + 1$.

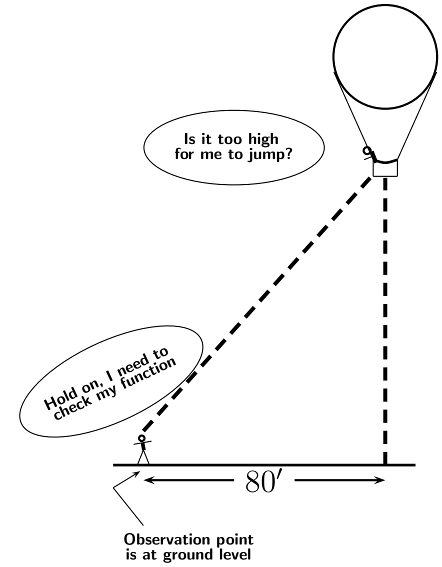
6. Let $f(x) = 3x - 5$ and $g(x) = \dfrac{x+5}{3}$.

   (a) Find $f[g(5)]$ and $g[f(-2)]$. Give your answers clearly, using this notation.

   (b) Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Again, make your answers clear.

   (c) Do you notice anything unusual here?

7. *This exercise illustrates how compositions of functions are used in applications.* A hot air balloon is released, and it rises at a rate of 5 feet per second. An observation point is situated on the ground, 80 feet from the point where the balloon is released, as shown to the right. (Picture courtesy Dr. Jim Fischer, OIT.)



   (a) Find the height h of the balloon as a function of time.

   (b) Find the distance d from the observation point to the balloon as a function of the height h.

   (c) Substitute your result from (a) into your function from (b) to determine the distance d as a function of t. What you are really doing here is finding $(d \circ h)(t)$.

5.3 Inverse Functions

Performance Criteria:

5. (f) Use composition to determine whether two functions are inverses of each other.

   (g) Compute the inverse of a function.

Introduction

In Exercise 6 of Section 5.2 you found that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for the functions $f(x) = 3x - 5$ and $g(x) = \dfrac{x+5}{3}$. This seems a bit unusual; what does it mean? Remember that $f \circ g$ is a single function that when applied to x is equivalent to performing g on x and then performing f on the result. The fact that $(f \circ g)(x) = x$ tells us that the "input" x is unchanged by the action of g followed by f. Clearly both f and g *DO NOT* leave their inputs unchanged, so what must be happening is that f "undoes" whatever g "does." Since we also have $(g \circ f)(x) = x$, g undoes f as well.

When this situation occurs we say that f and g are **inverse functions** of each other. In order for the computation $f[g(x)]$ to make sense we require that the range (outputs) of g coincide with the domain (inputs) of f; similarly, we also ask that the range of f coincide with the domain of g. To summarize this symbolically, we will write $f : A \to B$ to indicate that the domain of f is set A and its range is set B. We then have

Definition of Inverse Functions

Two functions $f : A \to B$ and $g : B \to A$ are inverses of each other if both

$(f \circ g)(x) = x$ for all x in B and $(g \circ f)(x) = x$ for all x in A.

The bit about sets A and B is a little technical (and has been oversimplified some here). We will sidestep that issue for now by considering only functions for which both the domains and ranges are all real numbers. In that case we can merely check to see that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

◇ **Example 5.3(a):** Are the functions $f(x) = 5x + 1$ and $g(x) = \dfrac{x-1}{5}$ inverses of each other?

Solution: We see that

$$(f \circ g)(x) = f[g(x)] = f\left[\frac{x-1}{5}\right] = 5\left(\frac{x-1}{5}\right) + 1 = (x-1) + 1 = x$$

and

$$(g \circ f)(x) = g[f(x)] = g[5x+1] = \frac{(5x+1)-1}{5} = \frac{5x}{5} = x,$$

so f and g are inverse functions of each other.

◇ **Example 5.3(b):** Are $f(x) = 3x+6$ and $g(x) = \frac{1}{3}x - 6$ inverses of each other?

Solution: Here we have

$$(f \circ g)(x) = f[g(x)] = f\left[\frac{1}{3}x - 6\right] = 3\left(\frac{1}{3}x - 6\right) + 6 = (x-18)+6 = x - 12.$$

Since $(f \circ g)(x) \neq x$, f and g are not inverses.

Note that if *EITHER* of $(f \circ g)(x)$ or $(g \circ f)(x)$ is not equal to x, the functions f and g are not inverses. That is why there was no need to check $(g \circ f)(x)$ in the previous example.

IMPORTANT: Usually if two functions are inverses we will not use f and g for their names. For a function f, we use the notation f^{-1} for its inverse. Thus, for the function $f(x) = 5x+1$ from Example 5.3(a) we have $f^{-1}(x) = \dfrac{x-1}{5}$.

There are a couple of questions you might have at this point:

- Do all functions have inverses?
- If I know a function f, how do I find its inverse function f^{-1}?

The answer to the first question is "NO" and we will discuss this a little more in the next section. First let's look at how we find the inverse of a function, assuming that it does have an inverse.

Finding Inverse Functions

Let's think carefully about the function $f(x) = 3x+2$, and how it "processes" an input to give an output. The first thing it does to an input is multiply it by three, *then* it adds two. The inverse must reverse this process, *including the order*. Think about dressing your feet in the morning - socks on first, then shoes. To undo this in the evening it is not socks off, then shoes off! You must not only reverse the processes of putting on both socks and shoes, but you must reverse the order as well. So to undo the function f one must first subtract two, then divide the result by three. The equation for f^{-1} is then $f^{-1}(x) = \dfrac{x-2}{3}$.

◇ **Example 5.3(c):** Find the inverse of $g(x) = \dfrac{x-3}{4}$.

Solution: If we were trying to find $g(2)$ we would start with two, subtract three and *then* divide by four. To invert this process we would first multiply by four, then add three. Therefore $g^{-1}(x) = 4x+3$.

- **Example 5.3(d):** Find the inverse of $h(x) = 4x^5 - 1$.

  Solution: To find $h(x)$ for a given value of x, one would first take that value to the fifth power, then multiply the result by four and, finally, subtract one. To invert those operations *and* the order in which they were done, we should first add one, divide that result by four, then take the fifth root of all that. The inverse function would then be $h^{-1}(x) = \sqrt[5]{\dfrac{x+1}{4}}$.

Section 5.3 Exercises To Solutions

1. Use the definition of inverse functions to determine whether each of the pairs of functions f and g are inverses of each other. (This means see if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You should *NOT* find the inverse of either function.)

   (a) $f(x) = 3x + 2, \; g(x) = \dfrac{x-2}{3}$ (b) $f(x) = 5x - 9, \; g(x) = \frac{1}{5}x + 9$

   (c) $f(x) = \sqrt[3]{\dfrac{x+1}{2}}, \; g(x) = 2x^3 - 1$ (d) $f(x) = \frac{3}{4}x - 3, \; g(x) = \frac{4}{3}x + 4$

2. The following functions are all invertible. Find the inverse of each, and give it using correct notation.

   (a) $g(x) = \dfrac{2x-1}{5}$ (b) $f(x) = 5x - 2$ (c) $h(x) = 2\sqrt[3]{x-1}$

   (d) $f(x) = \dfrac{x+3}{2}$ (e) $h(x) = x^3 + 4$ (f) $g(x) = \left(\dfrac{x-1}{7}\right)^5$

3. Find the inverse function for each of the following variations on 2(c).

   (a) $f(x) = \sqrt[3]{2x-1}$ (b) $g(x) = \sqrt[3]{2x} - 1$ (c) $h(x) = 2\sqrt[3]{x} - 1$

4. The equation $F = \frac{9}{5}C + 32$ allows us to change a Celsius temperature into Fahrenheit.

   (a) Convert $25°$C to a Fahrenheit temperature.

   (b) Solve the equation $F = \frac{9}{5}C + 32$ for C.

   (c) Freezing is $32°$F, or $0°$C. Use this to check your answer to (b).

   (d) Take your answer to (a) and use your answer to (b) to convert it back into a Celsius temperature.

   (e) What is the relationship between the function $F = \frac{9}{5}C + 32$ and your answer to (b)?

5. For each given function, find functions f, g and h for which $y = (f \circ g \circ h)(x)$.

   (a) $y = \left(\dfrac{x+2}{7}\right)^5$ (b) $y = \sqrt[3]{2x-1}$

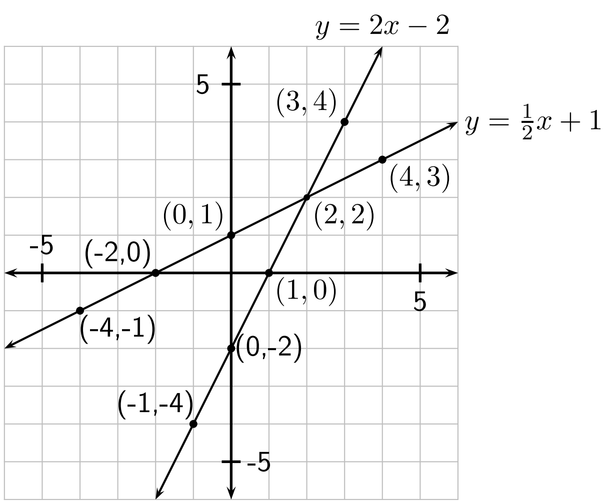
5.4 More on Inverse Functions

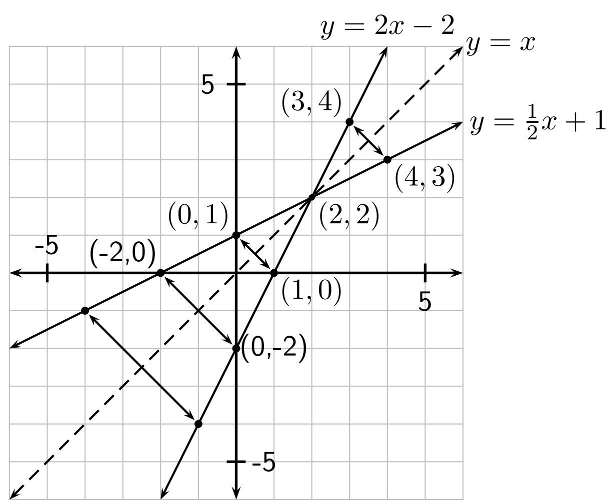
Performance Criteria:

5. (h) Given the graph of a function, draw or identify the graph of its inverse.
   (i) Determine whether a function is one-to-one based on its graph.
   (j) When necessary, restrict the domain of a function so that it has an inverse. Give the inverse and its domain.

Graphs of Inverse Functions

There is a special relationship between the graph of a function and the graph of its inverse. It can easily be shown that $y = \frac{1}{2}x + 1$ and $y = 2x - 2$ are inverses. We have graphed both functions on the same coordinate grid below and to the left. In addition we have labeled several points on each line with their coordinates. *Note that for each point* (a, b) *on the graph of* $y = \frac{1}{2}x + 1$, *the point* (b, a) *lies on the graph of* $y = 2x - 2$. On the graph below and to the right the line $y = x$ has been added, and the geometric relationship between these points with switched coordinates is shown.



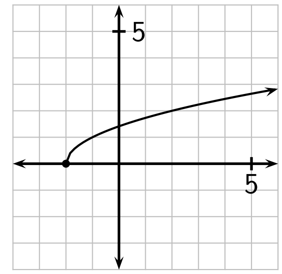


What we can see here is that for every point on the graph of a function that has an inverse (we say the function is **invertible**), there is a point on the inverse function the directly across the line $y = x$, and the same distance away from the line. We say the two graphs are **symmetric with respect to the line** $y = x$. This can be summarized as follows:

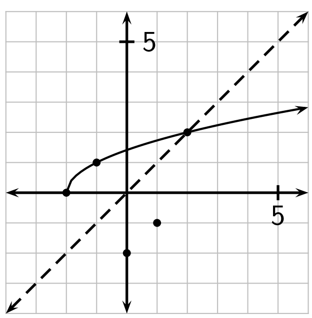
Graphs of Inverse Functions

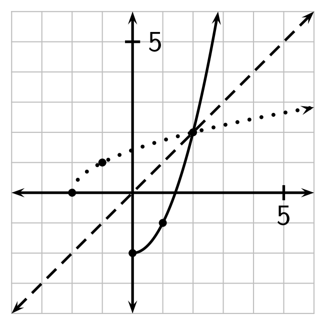
For an invertible function f, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

◇ **Example 5.4(a):** The function $y = \sqrt{x+2}$ is invertible, and its graph is shown to the right. (Note that it is just the graph of $y = \sqrt{x}$ shifted two units to the left, since the addition of two takes place before the root acts.) Draw another graph showing $y = \sqrt{x+2}$ as a dashed curve and, on the same grid, graph the inverse function.



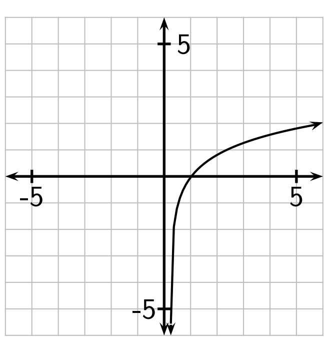
Solution: There are a couple of ways to approach this. Those with good visualization skills might just put in the line $y = x$ and simply draw the reflection of the graph over that line. Let's do that, but with a little help. First we note that the points $(-2, 0)$, $(-1, 1)$ and $(2, 2)$ are on the graph of the original function $y = \sqrt{x+2}$. This means that $(0, -2)$ and $(1, -1)$ are on the graph of the inverse, and $(2, 2)$ is on the graph of both the original function and the inverse. These points and the line $y = x$ are shown on the graph below and to the left. On the graph below and to the right, the new points have been used to draw the graph of the inverse (the solid curve) of the original function (shown as the dotted curve).

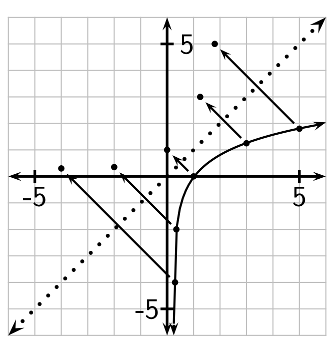


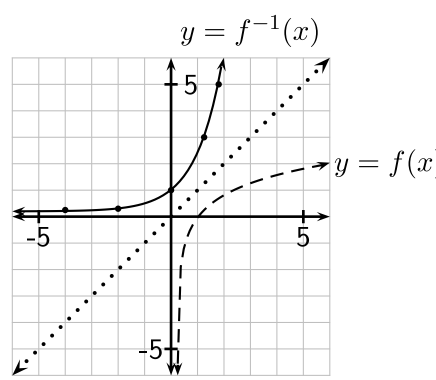


◇ **Example 5.4(b):** The graph of a function f is drawn on the grid below and to the left. Draw another graph, of the inverse function f^{-1}.

Solution: The middle grid below shows the reflection of several points on the graph of f across the line $y = x$. The right hand grid below shows how those points are connected to give the graph of the inverse function (the solid curve).







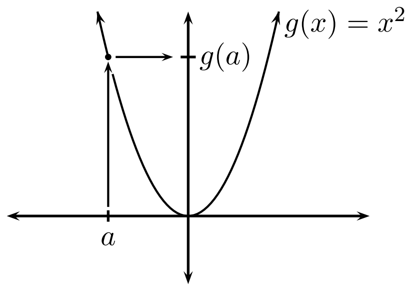
One-To-One Functions

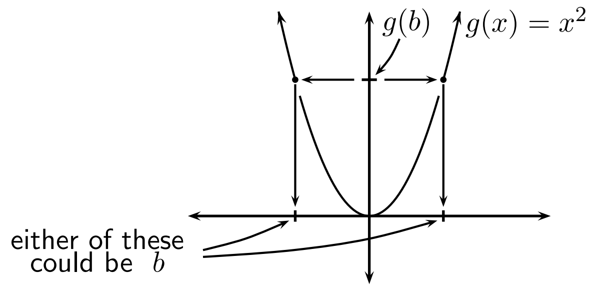
Consider the two functions $g(x) = x^2$ and $h(x) = \sqrt{x}$, and note that $h[g(5)] = h[25] = 5$. This seems to indicate that h is the inverse of g. However, $h[g(-3)] = h[9] = 3$, so maybe h is not the inverse of g! What is going on here? Well, for any function f that has an inverse f^{-1}, the inverse is suppose to take an output of the original function as its input and give out the input of the original function. That is,

$$\text{if } f(a) = b, \qquad \text{then} \qquad f^{-1}(b) = a.$$

For our two functions $g(-3) = 9$ but $h(9) = 3$, so they are (sort of) not inverses.

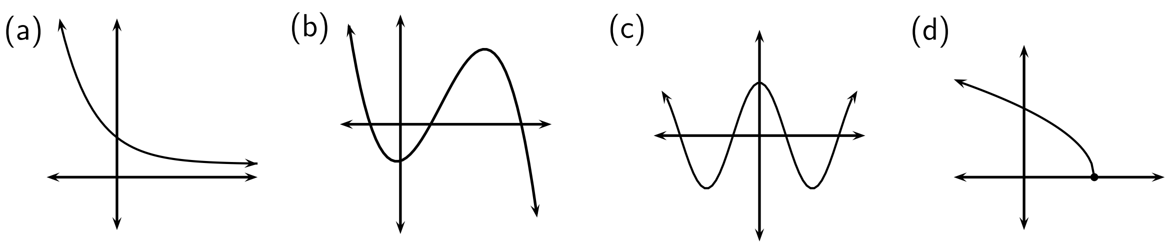
Let's look at this graphically. The graph of the function $g(x) = x^2$ is shown in both graphs below. On the graph to the left we see how to take a given input and obtain its output. The graph to the right shows how we would find the input(s) that give a particular output; note that one output comes from two different inputs.



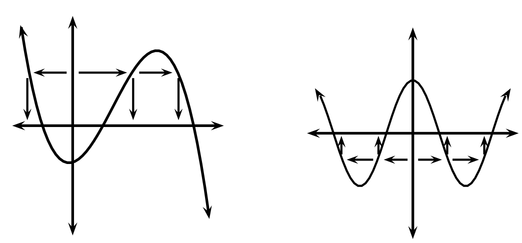


The problem here is that g is what we will call *two-to-one* for all values except zero. This means that two different inputs go to each output, so if we know an output of g, we cannot determine for certain what the input was that gave that output! For that reason $g(x) = x^2$ *is not invertible*. For a function to be invertible it is necessary for each output to have come from just one input. Such a function is called a **one-to-one** function. Graphically, no y value can have come from more than one x value in the way that it happened in the graph above and to the right.

◇ **Example 5.4(c):** The graphs of four functions are shown below. Determine which functions are one-to-one. For those that are not, indicate a y value that comes from more than one x value in the way shown with $g(x) = x^2$ above and to the right.



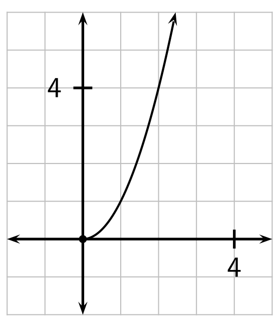
Solution: The functions shown in (b) and (c) are not one-to-one, as can be seen to the right.

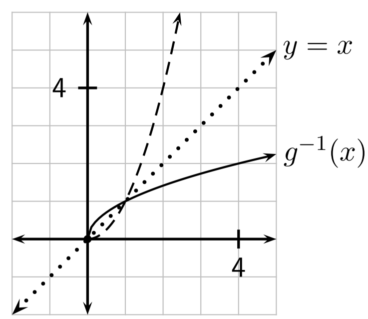


Let's get to the point now:

> **Invertibility of a Function**
>
> A function $f: A \to B$ is invertible if, and only if, f is one-to-one for all x in A.

On many occasions we want a function $f: A \to B$ to have an inverse, but the function isn't one-to-one. The key is to not let the set A be the entire domain of the function f; when we do this we say we are **restricting** the domain of the function. For example, if we consider the function $g(x) = x^2$ for only the values of x in the interval $[0, \infty)$, the graph of the function becomes the one shown to the left below. Clearly that function is one-to-one, so it has an inverse; the inverse is graphed on the grid below and to the right. As you might guess from the appearance of the graph, $g^{-1}(x) = \sqrt{x}$.

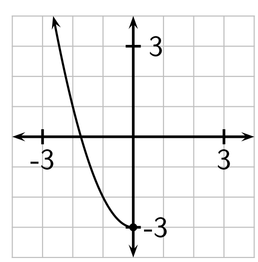




The next example combines this idea with the method we saw previously for finding the inverse of a function.

◇ **Example 5.4(d):** The graph below and to the right is for the function $f(x) = x^2 - 3$, $x \leq 0$. Find the inverse function.

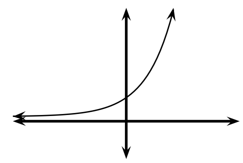
Solution: The function $f(x) = x^2 - 3$ first squares a number and then subtracts three. The inverse must first add three, then take either the positive or negative square root (we'll determine which later). So $f^{-1}(x) = \pm\sqrt{x+3}$ for now. We see that the point $(-1, -2)$ is on the graph of f, so $(-2, -1)$ must be on the graph of f^{-1}. Stated another way, $f^{-1}(-2)$ must be -1, which means that $f^{-1}(x) = -\sqrt{x+3}$.



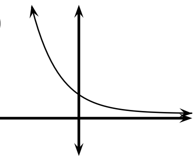
Section 5.4 Exercises

To Solutions

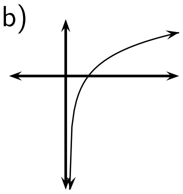
1. Look at the graph of f, shown to the right. Which of the graphs below is the graph of $f^{-1}(x)$?



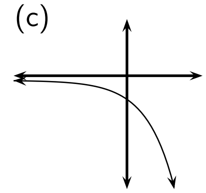
(a)



(b)



(c)



2. Do the following for each of the function/inverse function pairs below.

   - Graph the function and the line $y = x$ together with *Desmos*.
   - Sketch the graph on your paper and, on the same grid, graph what you think the graph of the inverse function would look like.
   - Plot the inverse function with *Desmos* (on the same grid with the other two graphs) and use the result to check your graph. For part (b), recall that $\sqrt[3]{x} = x^{\frac{1}{3}}$. *We have not yet worked with the functions in parts* (c) *and* (d), *but you should still be able to draw graphs of the inverse functions. Use the underline* (_) *to get a subscript.*

   (a) $f(x) = 2x - 3, \quad f^{-1}(x) = \dfrac{x+3}{2}$

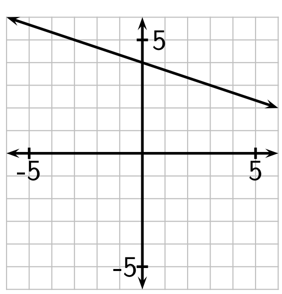
   (b) $g(x) = \sqrt[3]{x} + 1, \quad g^{-1}(x) = (x-1)^3$

   (c) $h(x) = 2^x, \quad h^{-1}(x) = \log_2 x$

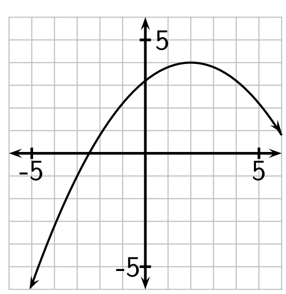
   (d) $f(x) = e^{-x}, \quad f^{-1}(x) = -\ln x$

3. Determine which of the functions below are one-to-one. For those that are, draw the graph of the function as a dashed curve on separate paper and, on the same grid, draw the graph of the inverse function as a solid curve. For those that are not, give a mathematical statement of the form $f(a) = f(b) = c$ for specific values of a, b and c.

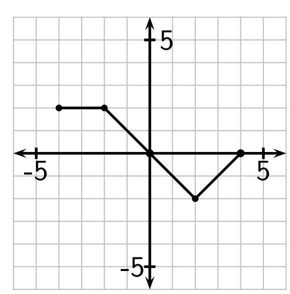
(a)



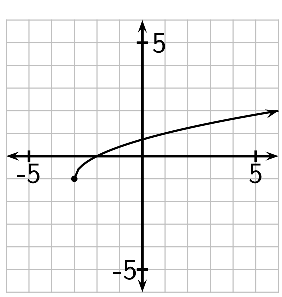
(b)



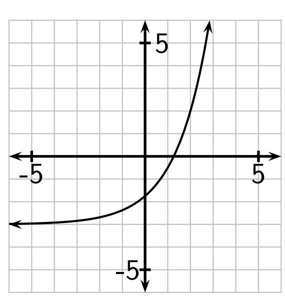
(c)



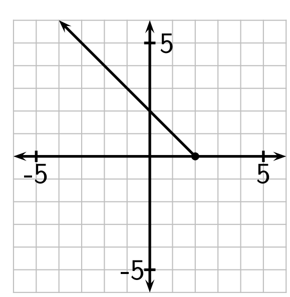
(d)



(e)



(f)



4. (a) Recall that the graph of $f(x) = (x-3)^2$ is a shift of the graph of $y = x^2$. Do you remember which way and how far? Refer to Section 2.6 if you need to. Sketch the graph, then check your answer by graphing the function with *Desmos*.

   (b) The graph indicates that f is not a one-to-one function. How would we restrict the domain in order to get just the right side of the parabola? Add $\{\ restriction\ \}$ to the *Desmos* command line for graphing f to see the graph with the restriction.

   (c) On a new grid, sketch the restricted function. On the same grid, sketch the graph of the inverse function f^{-1}.

   (d) Compute the inverse in the way that you learned in Section 5.3. Graph the inverse function with *Desmos*, and see if the graph matches your answer to (c). If it doesn't, figure out what is wrong and fix it!

5. (a) Sketch the left half of graph of $f(x) = (x-3)^2$. What restriction is this? Graph with *Desmos* to check your answer.

   (b) Sketch the graph of the inverse function on the same grid.

   (c) Determine algebraically the inverse function. Graph it with *Desmos* to check your answer - if it does not match your answer to (b), determine which is incorrect and fix it.

5.5 Transformations of Functions

Performance Criteria:

5. (k) Identify the graphs of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

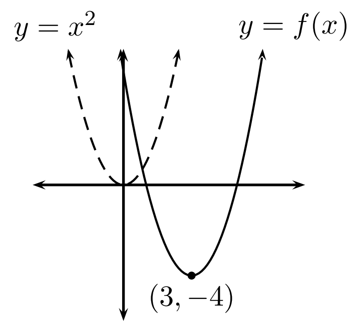
   (l) Given the graph of a function, sketch or identify various transformations of the function.

Some of the ides in this section were seen previously in Section 2.6; we will repeat those ideas here, reinforcing them with things you saw in Sections 3.2 and 5.2.

Introduction, Shifts of Graphs of Functions

We know that the graph of the function

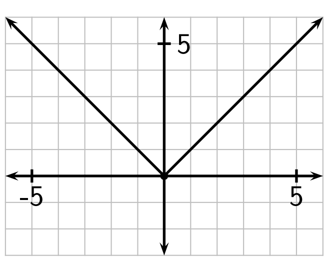
$$f(x) = (x-3)^2 - 4$$



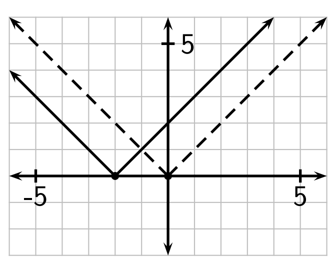
is a parabola opening upward, with vertex at $(3, -4)$. We also know that it has the exact same shape as $y = x^2$. The graph to the right shows f graphed on the same axes as $y = x^2$. One can see that the graph of f can be obtained by "sliding" the graph of $y = x^2$ three units to the right and four units down.

We should note that the function f is really a composition of three functions; first we subtract 3, then we square, then we subtract four. Here we will want to consider the squaring to be the "main function," with the actions of subtracting three and subtracting four occurring before and after the main function, respectively.

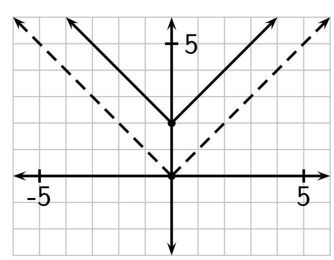
Let's now consider the absolute value function $y = |x|$, whose graph is shown below and to the left, and the two functions $g(x) = |x+2|$ and $h(x) = |x| + 2$. Note the difference between the two functions. With g , we add two *before* the main function of absolute value, and with h we add two *after* the absolute value. The graphs of g and h are shown on the middle and right hand grids below. For both, the graph of $y = |x|$ is shown with dashed lines.



$y = |x|$



$g(x) = |x+2|$



$h(x) = |x| + 2$

What we see here is that when we add two *before* applying the main function of absolute value, the graph of the function is that of $y = |x|$ shifted two units *to the left*, or in the negative direction. This may seem a bit counterintuitive. The right way to think about it is that the value

$x = -2$ results in $|0| = 0$, which is where the "vertex" of the absolute value function occurs. (The term vertex is generally reserved for parabolas, but it certainly would make sense to think of the point of the absolute value function as sort of a vertex as well.) When we add two *after* the absolute value, the result is that the graph is shifted up by two. What is really happening is that the y values are obtained by first computing $|x|$, then two units are added to each y, moving the whole graph up by two.

The following summarizes what you have been seeing. When reading it, think of the function f as being something like squaring, or absolute value.

Shifts of Graphs:

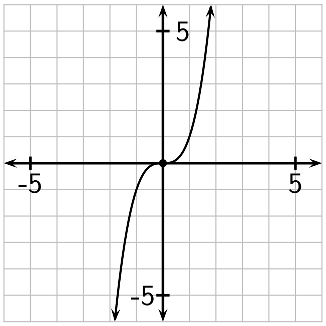
Let a and b be *positive* constants and let $f(x)$ be a function.

- The graph of $y = f(x+a)$ is the graph of $y = f(x)$ shifted a units to the left. That is, what happened at zero for $y = f(x)$ happens at $x = -a$ for $y = f(x+a)$.
- The graph of $y = f(x-a)$ is the graph of $y = f(x)$ shifted a units to the right. That is, what happened at zero for $y = f(x)$ happens at $x = a$ for $y = f(x-a)$.
- The graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up by b units.
- The graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down by b units.

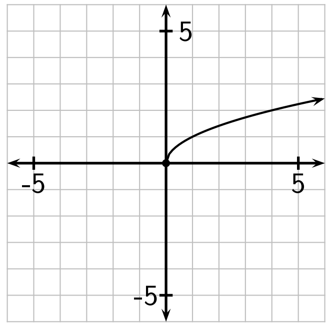
Note that when we add or subtract the value a *before* the function acts, the shift is in the x direction, and the direction of the shift is the opposite of what our intuition tells us should happen. When we add or subtract b *after* the function acts, the shift is in the y direction and the direction of the shift is up for positive and down for negative, as our intuition tells us it should be.

Some Basic Functions

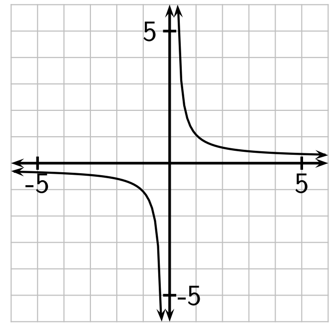
Before continuing, we need to pause to consider the graphs of some common functions. You should by now be quite familiar with the graph of $y = x^2$, and we just saw the graph of $y = |x|$. Three other graphs you should have committed to memory are those of $y = x^3$, $y = \sqrt{x}$ and $y = \frac{1}{x}$, all of which are shown at the top of the next page.



$y = x^3$



$y = \sqrt{x}$

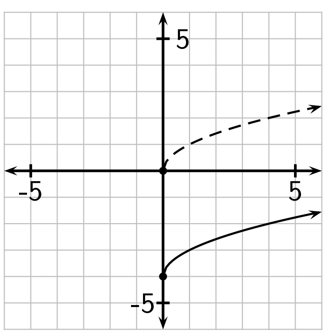


$y = \dfrac{1}{x}$

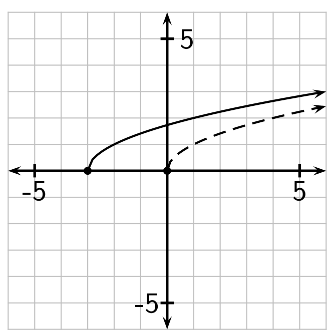
We will use the graphs of $y = x^2$, $y = |x|$, $y = x^3$ and $y = \sqrt{x}$ to demonstrate the principles of this section. Using these basic functions, we can now look at a few more examples of shifts of functions.

◇ **Example 5.5(a):** Sketch the graphs of $y = \sqrt{x} - 4$ and $y = \sqrt{x+3}$, putting the graph of $y = \sqrt{x}$ on the same grid with each.

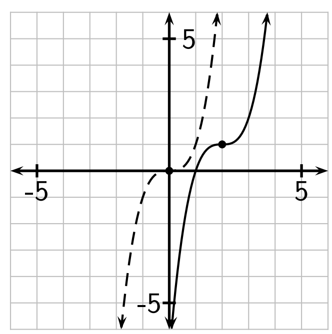
Solution: For the function $y = \sqrt{x} - 4$, the subtraction of four is taking place *after* the square root, so it will shift the graph of $y = \sqrt{x}$ vertically. Vertical shifts are as they appear, so the shift will be down. The result is shown in the graph below and to the left. $y = \sqrt{x} - 4$ is the solid curve, $y = \sqrt{x}$ is the dashed. For $y = \sqrt{x+3}$, the addition of three is *before* the square root, so the shift will be in the x-direction and opposite of what the sign indicates, so to the left. Again, this is because the square root will be acting on zero at $x = -3$. The graph of this function is on the middle graph below, as a solid curve. Again, the dashed curve is $y = \sqrt{x}$.



$y = \sqrt{x} - 4$



$y = \sqrt{x+3}$



$y = (x-2)^3 + 1$

◇ **Example 5.5(b):** Sketch the graph of $y = (x-2)^3 + 1$ on the same grid with the graph of $y = x^3$.

Solution: Here the function $y = x^3$ is composed with two functions, a subtraction of two before cubing, and an addition of one after cubing. The subtraction of two before the cubing causes a shift to the right of two, and the addition of one after causes a shift of one upward. The final result is seen in the right hand graph above.

Stretches and Shrinks of Graphs of Functions

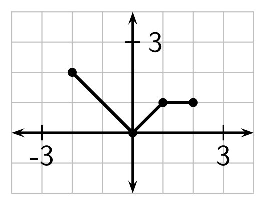
Next we look at the effect of multiplying, which causes the graph of a function to "stretch" or "shrink" in either the x- or y-direction, again depending on whether it occurs before or after the function acts.

> **Stretches and Shrinks of Graphs:**
>
> Let a and b be *positive* constants and let $f(x)$ be a function.
>
> - If $a > 1$, the graph of $y = f(ax)$ is the graph of $y = f(x)$ shrunk by a factor of $\frac{1}{a}$ in the x direction, toward the y-axis. If $a < 1$, the graph of $y = f(ax)$ is the graph of $y = f(x)$ stretched by a factor of $\frac{1}{a}$ in the x direction, away from the y-axis.
> - If $b > 1$, the graph of $y = bf(x)$ is the graph of $y = f(x)$ stretched by a factor of b in the y direction, away from the x-axis. If $b < 1$, the graph of $y = bf(x)$ is the graph of $y = f(x)$ shrunk by a factor of b in the y direction, toward the x-axis.

Here we see again the phenomenon that an algebraic change before f acts causes a change in the graph in the x direction, but the change is again the opposite of what we think intuitively should happen. An algebraic change *after* f acts results in a change in the graph in the vertical direction, and in the way that our intuition tells us it should happen.

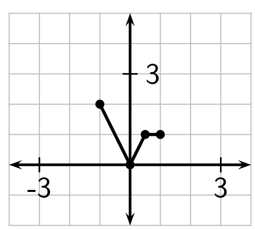
For all of the basic functions that we are currently familiar with, the form $y = f(ax)$ can always be converted to $y = bf(x)$. The form $y = f(ax)$ is quite important in trigonometry, however, so let's take a look at it here. There is no need to have an equation in order to illustrate the principles we are talking about, though, so we'll just go with a graph.

◇ **Example 5.5(c):** The graph of a function $y = f(x)$ is shown to the right. Draw the graphs of $y = f(2x)$ and $y = 2f(x)$ on separate graphs.

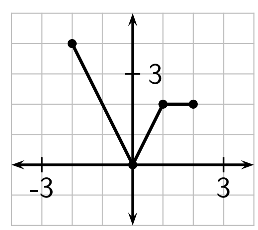


$y = f(x)$

Solution: For the function $y = f(2x)$, the multiplication by two is taking place before the function f acts, so the effect is in the horizontal, or x, direction. Since all horizontal effects are the opposite of what our intuition tells us, this means the graph shrinks toward the y-axis by a factor of one half. This is shown in the first graph to the right.



$y = f(2x)$



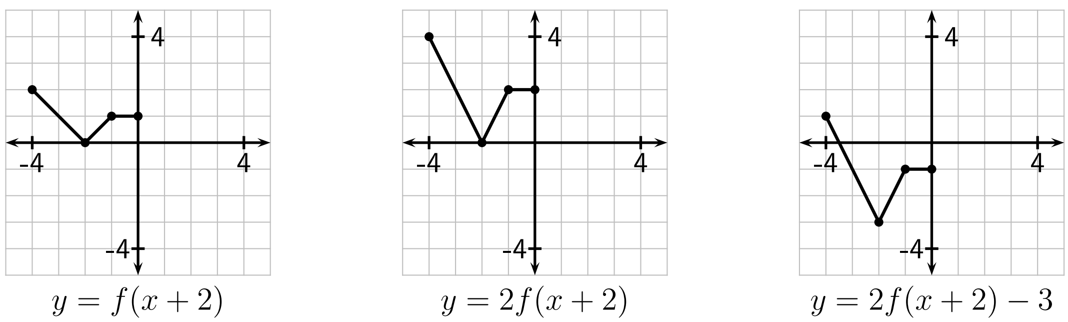
$y = 2f(x)$

For the function $y = 2f(x)$, the multiplication takes place after f acts. Therefore the effect is vertical and exactly as it seems - the graph is stretched vertically away from the x-axis by a factor of two, as shown in the second graph above and to the right.

We can combine stretches or shrinks with shifts:

◇ **Example 5.5(d):** For the function f from the previous exercise, sketch the graph of $y = 2f(x+2) - 3$.

Solution: For function like this, we want to think about the order of operations that are applied to x. In this case, we add two before f acts, so that shifts the function two units to the left, as shown in the left hand graph below. Next f acts, then we multiply by two *after* f has acted, so this causes a stretch by a factor of two away from the x-axis. This is shown on the middle graph below. Finally we subtract three, resulting in a downward shift by three units. The final graph is shown below and to the right.



Reflections (Flips) of Graphs

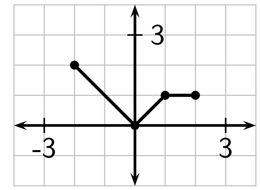
When we multiply by a negative, the graph of the function gets flipped over the x- or y-axis, depending on whether the multiplication takes place before or after the function acts.

Reflections of Graphs:

For a function $y = f(x)$ and *negative* numbers a and b, we have the following:

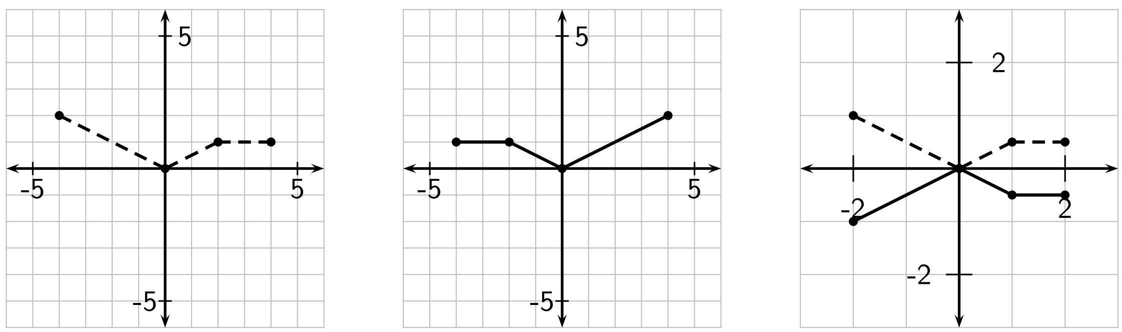
- The effect of a in $y = f(ax)$ is the same as when a is positive, except that the graph not only stretches or shrinks away from or toward the y-axis, but it "flips" over the axis as well. We say that the graph **reflects** across the y-axis.
- The effect of b in $y = bf(x)$ is the same as when b is positive, but in this case there is a reflection across the x-axis.

◇ **Example 5.5(e):** The graph of a function $y = f(x)$ is shown to the right. Draw the graphs of $y = f(-\frac{1}{2}x)$ and $y = -\frac{1}{2}f(x)$ on separate graphs.



$y = f(x)$

Solution: For the function $y = f(-\frac{1}{2}x)$, the multiplication by one-half is taking place before the function f acts, so the effect is in the horizontal, or x, direction. Since all horizontal effects are the opposite of what our intuition tells us, this means the graph stretches away from the y-axis by a factor of two. This is shown by the dashed graph below and to the left. But the fact that we are really multiplying by *negative* one-half means that the graph also reflects in the x-direction, across the y-axis, so the final graph is the solid one shown in the middle below.

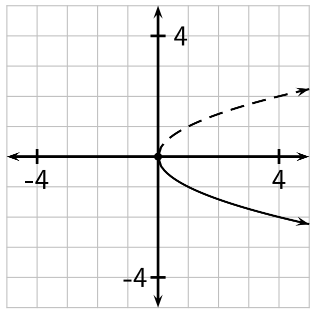


$y = f(\frac{1}{2}x)$ $\qquad$ $y = f(-\frac{1}{2}x)$ $\qquad$ $y = -\frac{1}{2}f(x)$

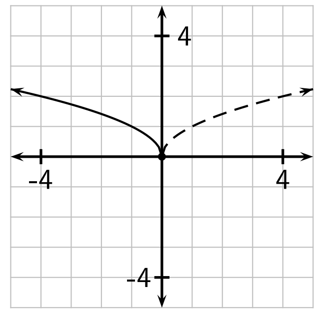
For the function $y = -\frac{1}{2}f(x)$, the multiplication takes place after f acts, so the graph shrinks to half its original distance from the x-axis, as shown by the dashed graph on the right grid above. (Note the change in scale.) But the multiplication by a negative causes the graph to reflect vertically, across the x-axis, so the final result is as shown by the solid graph on the right grid above.

- **Example 5.5(f):** Sketch the graph of $y = \sqrt{x}$ as a dashed curve. Then, on the same grid, graph $y = -\sqrt{x}$. Repeat this for $y = \sqrt{-x}$ and $y = \sqrt{-x} + 2$.

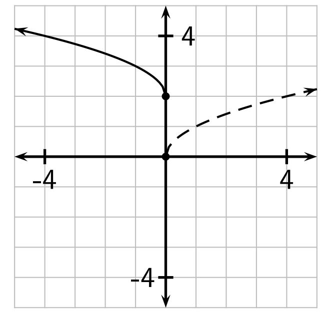
Solution: For $y = -\sqrt{x}$, the negative is applied after the function, so the reflection is vertical, over the x axis. This is shown on the left graph at the top of the next page. The reflection is horizontal, or over the y-axis, for $y = \sqrt{-x}$, as shown in the middle graph at the top of the next page.



$y = -\sqrt{x}$



$y = \sqrt{-x}$



$y = \sqrt{-x} + 2$

For the function $y = \sqrt{-x} + 2$, the graph first reflects across the y-axis, then moves up two units, as shown in the third graph above.

Conclusion

In this section we have seen how certain changes to a common function change the appearance of its graph. we can summarize the basic principles as follows:

Transformations of Graphs:

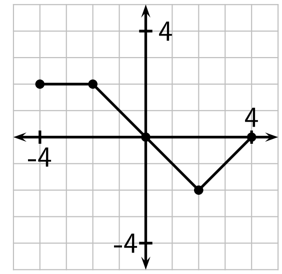
Variations of a function $f(x)$ result in transformations of the graph of $f(x)$ according to the following principles:

- Addition or subtraction cause shifts of the graph.
- Multiplication causes stretches or shrinks of the graph.
- Multiplication by a negative causes a reflection of the graph.
- All of the above affect the graph in the x-direction (horizontally) if they occur before the main function acts. They affect the graph in the y-direction (vertically) if they occur after the main function acts.
- Horizontal effects are the opposite of what they seem they should be, and vertical effects are as they seem they should be.

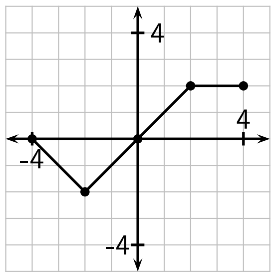
You've seen examples of all these things. You've also seen, in a few of the examples, how combinations of more than one of the actions affects the graph of the main function. There will be more of those in the exercises, and we'll look at one last example of such in a moment. One thing we won't do at this point is combine a multiplication and an addition (or subtraction) before the function acts. The effect of that is a bit more complicated to analyze than what we have looked at so far.

◇ **Example 5.5(g):** The graph of a function $y = g(x)$ is shown to the right. Sketch the graph of $y = \frac{1}{2}g(-x) - 3$.

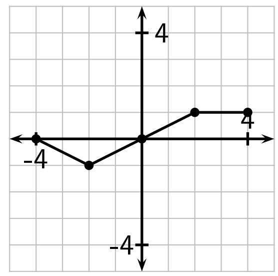
Solution: Here there are three actions taking place: multiplication by a negative one before g acts, and Multiplying by $\frac{1}{2}$ and subtraction three after g acts. We can get the desired graph by accumulating those three actions one at a time, as shown in the graphs at the top of the next page.



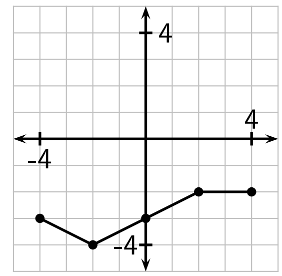
$y = g(x)$



$y = g(-x)$



$y = \frac{1}{2}g(-x)$



$y = \frac{1}{2}g(-x) - 3$

Section 5.5 Exercises

To Solutions

1. For each of the quadratic functions below,

   - sketch the graph of $y = x^2$ **accurately** on a coordinate grid,
   - **on the same grid**, sketch what you think the graph of the given function would look like,
   - graph $y = x^2$ and the given function together on your calculator to check your answer

   (a) $h(x) = (x+1)^2$ (b) $y = x^2 - 4$

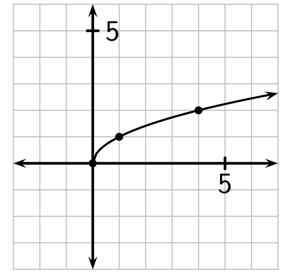
   (c) $g(x) = \frac{1}{4}(x+2)^2 - 1$ (d) $y = -2(x-4)^2 + 2$

2. For each of the following, graph $h(x) = x^3$ and the given function on the same graph. Check your answers using a graphing calculator or an online grapher.

   (a) $y = -x^3$ (b) $y = \frac{1}{2}x^3 + 4$ (c) $y = (x-1)^3$

   (d) $y = (x+3)^3 + 1$ (e) $y = -(x+2)^3$

3. The graph of $y = \sqrt{x}$ is shown to the right, with the three points $(0,0)$, $(1,1)$ and $(4,2)$ indicated with dots. For each of the following, sketch what you think the graph of each would look like, along with a dashed copy of the graph of $y = \sqrt{x}$. On the graph of the new function, indicate three points showing the new positions of the three previously mentioned points. Check your answers in the back, making sure to see that you have the correct points plotted.



$y = \sqrt{x}$

(a) $y = \sqrt{x-1}$ (b) $y = \sqrt{x} - 3$ (c) $y = 2\sqrt{x}$

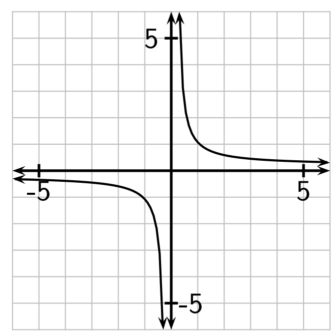
(d) $y = 2\sqrt{x} - 3$ (e) $y = \sqrt{-x} + 1$ (f) $y = -\sqrt{x-2} + 4$

4. For each of the following, graph $f(x) = \frac{1}{x}$ (its graph is shown below and to the right) and the given function on the same graph, with the graph of $f(x) = \frac{1}{x}$ dashed.

(a) $y = \dfrac{1}{x+3}$ (b) $y = \dfrac{1}{x} + 3$

(c) $y = -\dfrac{1}{x}$ (d) $y = \dfrac{1}{-x}$

(e) $y = -\dfrac{1}{x-2}$ (f) $y = \dfrac{1}{x+2} + 1$



$y = \dfrac{1}{x}$

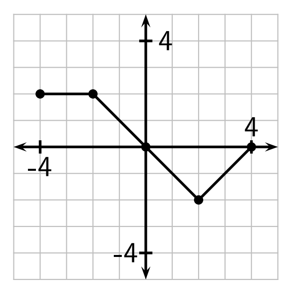
5. Consider again the function $y = g(x)$ from Example 5.2(g), shown below and to the right. Use it to graph each of the following.

(a) $y = 2g(x)$ (b) $y = g(-2x)$

(c) $y = g(x) - 2$ (d) $y = g(x-2)$

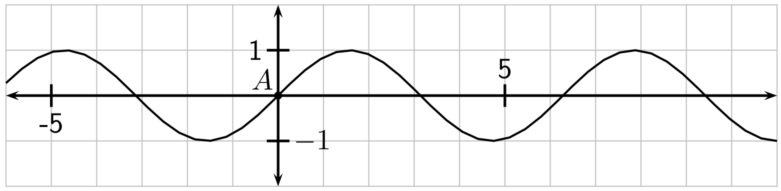
(e) $y = \frac{1}{2}g(x+3)$ (f) $y = g(\frac{1}{2}x) - 3$

(g) $y = -g(x-1) + 2$ (h) $y = -\frac{1}{2}g(x-3)$

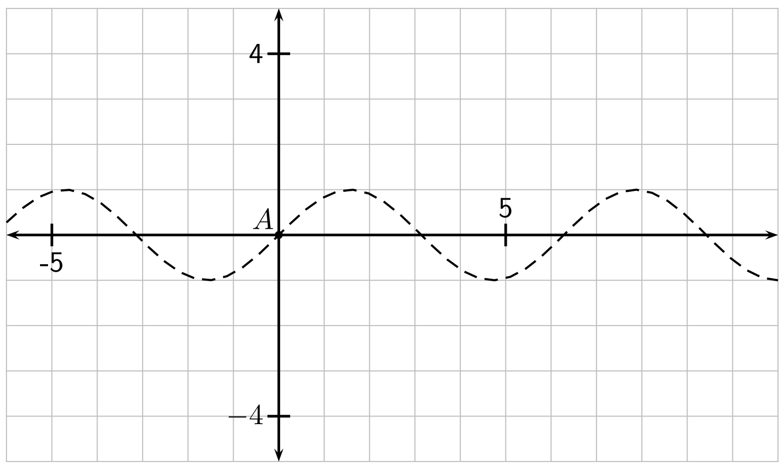


$y = g(x)$

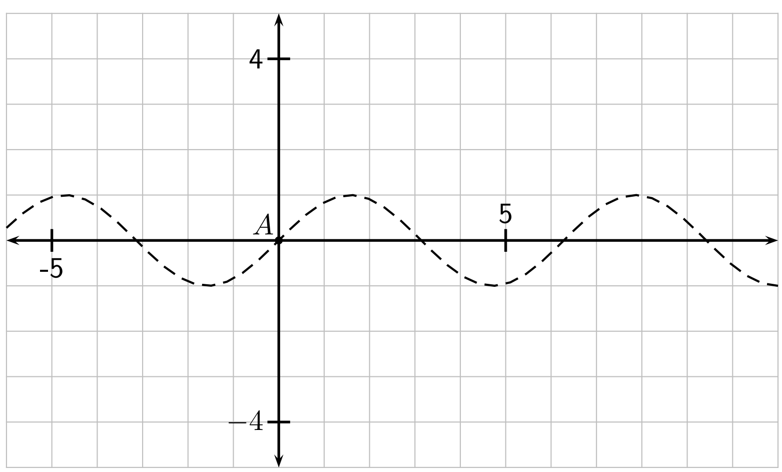
6. Some of you will go on to take trigonometry, in which you will study functions that are called *periodic*. This means several things; one characteristic of periodic functions is that their graphs consist of one portion repeated over and over. One of the most basic periodic functions is called the *sine function*, which is denoted $f(x) = \sin x$. Its graph is shown below. The point labeled A is just for reference when doing the exercises below.



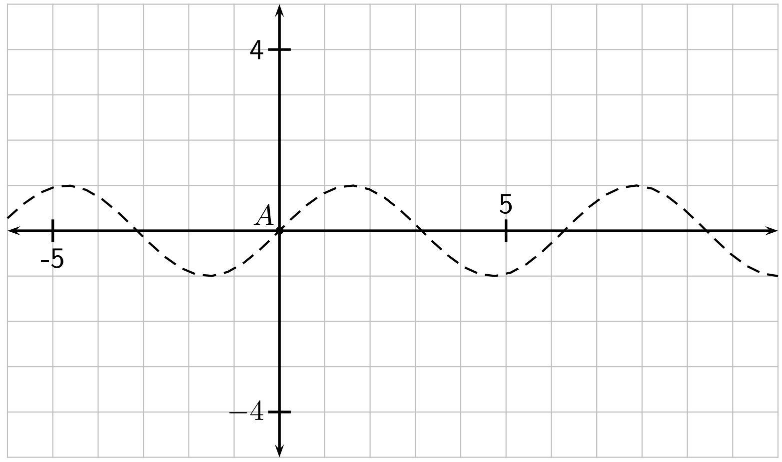
(a) Sketch the graph of $g(x) = \sin(x - 2)$ on the grid below. The dashed curve is the graph of $y = g(x)$. Label the new position of the point A from the graph above as A'.



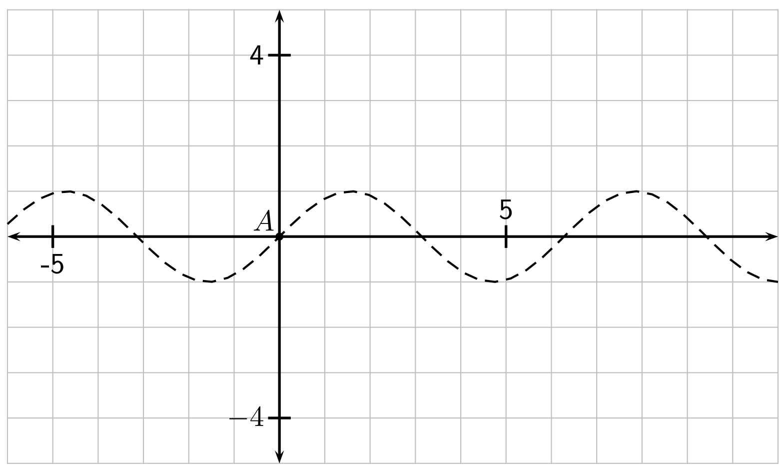
(b) Sketch the graph of $g(x) = \sin(x) - 2$ on the grid below.



(c) Sketch the graph of $h(x) = 3\sin x$ on the grid below.



(d) Sketch the graph of $y = 2\sin(x+1)$, again labeling the new position of A as A'. Again, check with your calculator.



Understanding how the graph of the sine function and other similar functions are shifted and stretched is a major part of trigonometry.

6 Exponential and Logarithmic Functions

Outcome/Performance Criteria:

6. Understand and apply exponential and logarithmic functions.

   (a) Evaluate exponential functions. Recognize or create the graphs of $y = a^x$, $y = a^{-x}$ for $0 < a < 1$ and $1 < a$.

   (b) Determine inputs of exponential functions by guessing and checking, and by using a graph.

   (c) Solve problems using doubling time or half-life.

   (d) Use exponential models of "real world" situations, and compound interest formulas, to solve problems.

   (e) Solve problems involving e.

   (f) Evaluate logarithms using the definition.

   (g) Change an exponential equation into logarithmic form and vice versa; use this to solve equations containing logarithms.

   (h) Use the inverse property of the natural logarithm and common logarithm to solve problems.

   (i) Use properties of logarithms to write a single logarithm as a linear combination of logarithms, or to write a linear combination of logarithms as a single logarithm.

   (j) Apply properties of logarithms to solve logarithm equations.

   (k) Solve an applied problem using a given exponential or logarithmic model.

   (l) Create an exponential model for a given situation.

Exponential functions are used to model a large number of physical phenomena in engineering and the sciences. Their applications are as varied as the discharging of a capacitor to the growth of a population, the elimination of a medication from the body to the computation of interest on a bank account. In this chapter we will see a number of such applications.

In many of our applications we will have equations containing exponential functions, and we will want to solve those equations. In some cases we can do that by simply using our previous knowledge (usually with the help of a calculator). In other cases, however, we will need to invert exponential functions. This is where logarithm functions come in - they are the inverses of exponential functions.

6.1 Exponential Functions

Performance Criteria:

6. (a) Evaluate exponential functions. Recognize or create the graphs of $y = a^x$, $y = a^{-x}$ for $0 < a < 1$ and $1 < a$.
   (b) Determine inputs of exponential functions by guessing and checking, and by using a graph.
   (c) Solve problems using doubling time or half-life.

Introduction to Exponential Functions

The simplest examples of exponential functions are functions of the form $y = a^x$, where $a > 0$ and $a \neq 1$, like $y = 2^x$. To better understand such functions, let's recall some facts about exponents:

- $a^0 = 1$ for any number $a \neq 0$. 0^0 is undefined.
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

◇ **Example 6.1(a):** For $f(x) = 4^x$, find values of the function for $x = 2,\ 3,\ 1,\ \frac{1}{2},\ 0,\ -1,\ -2,\ -\frac{1}{2}$ without using a calculator.

Solution: We know of course that

$$f(2) = 4^2 = 16, \qquad f(3) = 4^3 = 64, \qquad f(1) = 4^1 = 4.$$

From the above facts we also have

$$f(\tfrac{1}{2}) = 4^{\frac{1}{2}} = \sqrt{4} = 2, \qquad f(0) = 4^0 = 1, \qquad f(-1) = 4^{-1} = \frac{1}{4^1} = \tfrac{1}{4}$$

and

$$f(-2) = 4^{-2} = \frac{1}{4^2} = \tfrac{1}{16}, \qquad f(-\tfrac{1}{2}) = 4^{-\frac{1}{2}} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \tfrac{1}{2}$$

The reason we require $a > 0$ is that a^x is undefined for $x = \frac{1}{2}$ when a is negative and a^x is also undefined for negative values of x when a is zero.

We will want to be able to evaluate such functions using our calculators. This is done by using the carat (^) button or, if your calculator does not have such a button, the y^x button. You should try using your calculator to do the following computations yourself, making sure you get the results given.

◇ **Example 6.1(b):** For $f(x) = 5^x$, find $f(3.2)$ to the thousandth's place.

Solution: To evaluate this with a graphing calculator and some non-graphing calculators, simply enter $5\,\hat{}\,3.2$ and hit *ENTER*. The result (rounded) should be 172.466. With some non-graphing calculators you will need to use the y^x button. Enter $5\ y^x\ 3.2$ followed by $=$ or ENTER to get the value.

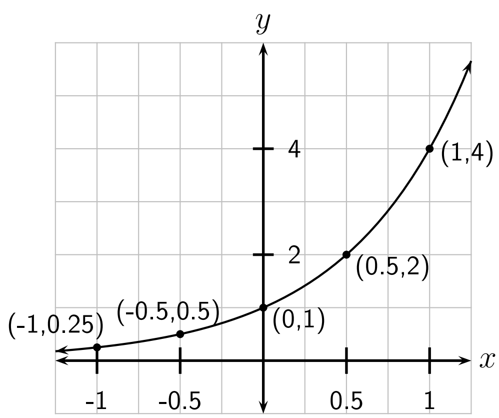
◇ **Example 6.1(c):** For $g(t) = 20(7)^{-0.08t}$, find $g(12)$ to the hundredth's place.

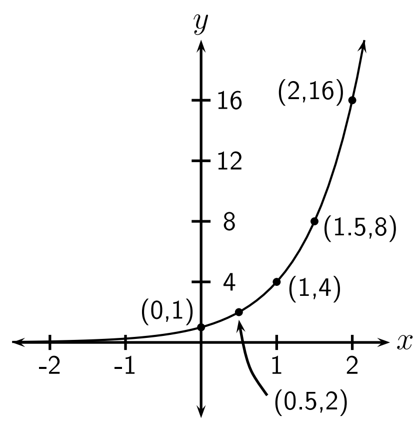
Solution: To get the correct result here you can simply enter $20 \times 7\,\hat{}\,(\,-0.08 \times 12\,)$ or $20 \times 7\ y^x\ (\,-0.08 \times 12\,)$. Note that you MUST use parentheses around the -0.08 and 12. The rounded result is 3.08.

Graphs of Exponential Functions

Suppose we wish to graph the exponential function $f(x) = 4^x$. Some values for this function were obtained in Example 6.1(a), and their decimal forms have been compiled in the table below and to the left and the ordered pairs ploteed on the graphs to the right of the table. Two graphs have been, so that we can see what is happening for values of x near zero and in the negatives, and then we see what happens as we get farther and farther from zero.

x	y
-2	0.0625
-1	0.25
-0.5	0.5
0	1
0.5	2
1	4
2	16



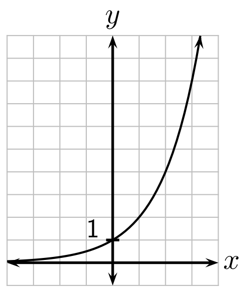


We get the point at $(1.5, 8)$ because $4^{1.5} = 4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$. Looking at these two graphs of the function $y = 4^x$ we make the following observations:

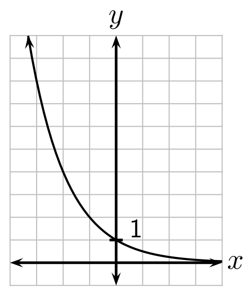
- The domain of the function is all real numbers.
- The range of the function is $(0, \infty)$, or $y > 0$. Even when the value of x is negative, $y = 4^x$ is positive. However, we can get y to be as close to zero as we want by simply taking x to be a negative number with large absolute value.
- The function is increasing for all values of x. For negative values of x the increase is very slow, but for x positive the increase is quite rapid.

The graphs illustrate what we call **exponential growth**, where the values of y increase at a faster and faster rate as x gets larger and larger.

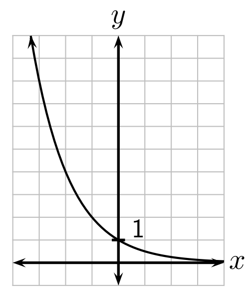
From the previous chapter we know that the graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ across the y-axis. We see this in the first two graphs below, of $y = 2^x$ and $y = 2^{-x}$. The function $y = 2^{-x}$ exhibits what we call **exponential decay**, in which the y values decrease as the x values increase, but the decrease in y is less rapid as x gets larger and larger.



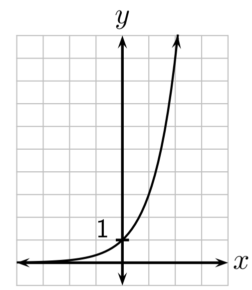
$y = 2^x$



$y = 2^{-x}$



$y = (\frac{1}{2})^x$



$y = 3^x$

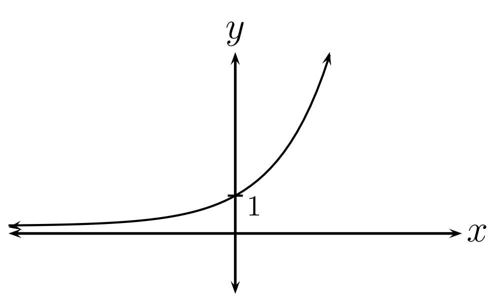
The third graph above is for $y = (\frac{1}{2})^x$, and we can see that the graph is the same as that of $y = 2^x$. This makes sense, because by the exponent rule $(x^a)^b = x^{ab}$ and the definition $x^{-n} = \dfrac{1}{x^n}$ we have $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$.

The last graph above is that of $y = 3^x$. We see that it is pretty much the same shape as the graph of $y = 2^x$, but the graph of $y = 3^x$ climbs more rapidly than that of $y = 2^x$ to the right of the y-axis, and the graph of $y = 3^x$ gets closer to zero than the graph of $y = 2^x$ as we go out in the negative direction.

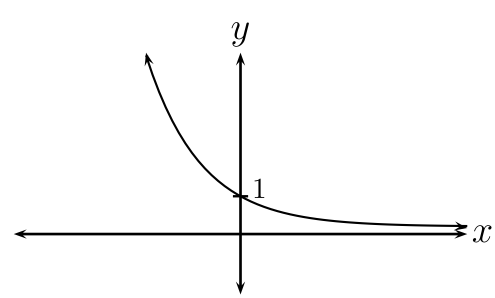
Finally we note that the y-intercepts of all of these are one, by the fact that $a^0 = 1$ if $a \neq 0$. All of these things can be summarized as follows:

- For $a > 1$, the function $y = a^x$ is increasing. When such a function describes a "real life" situation, we say that the situation is one of **exponential growth**. The larger a is, the more rapidly the function increases.
- For $0 < a < 1$, the function $y = a^x$ is decreasing, and the situation is one of **exponential decay**.
- Exponential decay also occurs with functions of the form $y = a^{-x}$, where $a > 1$.
- In all cases, the range of an exponential function is $y > 0$ (or $(0, \infty)$ using interval notation).

Graphically we have



$f(x) = a^x,\ (a > 1)$



$f(x) = a^x,\ (0 < a < 1)$
$f(x) = a^{-x},\ (a > 1)$

Since we have two ways to obtain exponential decay, we usually choose $a > 1$ and use the exponent $-x$ to obtain exponential decay. As a final note for this section, when graphing $f(x) = Ca^{kt}$, the y-intercept is C, rather than one.

Doubling Time and Half-Life

We conclude this section with two ideas (that are essentially the same) that arise commonly in the natural sciences and finance.

- One way to characterize the rate of exponential growth is to give the **doubling time** for the situation. This is the amount of time that it takes for the amount (usually a population or investment) to double in size. *This value remains the same at any point in the growth, not just at the beginning.*
- Exponential decay is often described in terms of its **half-life**, which is the amount of time that it takes for the amount to decrease to half the size. Like doubling time, the half-life is also the same at any point during the decay.

Let's look at a couple of examples.

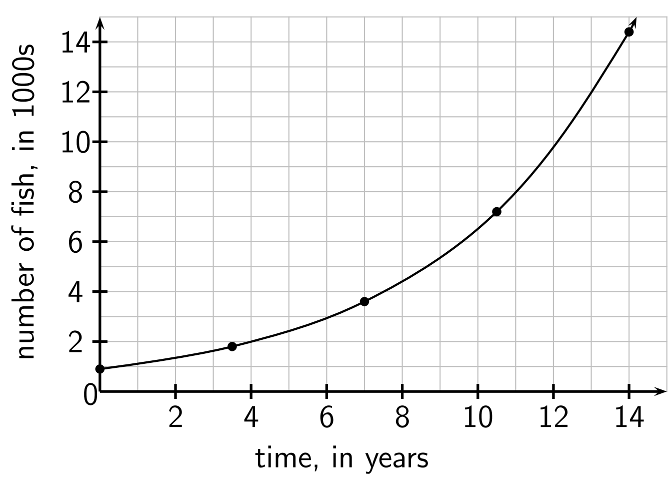
◇ **Example 6.1(d):** A new radioactive element, *Watermanium*, is discovered. It has a half-life of 1.5 days. Suppose that you find a small piece of this element, weighing 48 grams. How much will there be three days later?

Solution: Note that three days is two 1.5 day periods. After a day and a half, only half of the original amount, 24 grams, will be left. After another day and a half only half of those 24 grams will be left, so there will be 12 grams after three days.

◇ **Example 6.1(e):** The population of a certain kind of fish in a lake has a doubling time of 3.5 years, and it is estimated that there are currently about 900 fish in the lake. Construct a table showing the number of fish in the lake every 3.5 years from now (time zero) up until 14 years, and a graph showing the growth of the population of fish in the lake.

Solution: We begin our table by showing that when we begin (time zero) there are 900 fish. In 3.5 years the population doubles to $2 \times 900 = 1800$ fish. 3.5 years after that, or 7 years overall, the population doubles again to $2 \times 1800 = 3600$ fish. Continuing, we get the table below and to the left and, from it, the graph below and to the right.

time in years	number of fish
0.0	900
3.5	1800
7.0	3600
10.5	7200
14.0	14,400



Finding "Inputs" of Exponential Functions

Suppose that we wanted to solve the equation $37 = 2^x$. We might be tempted to take the square root of both sides, but $\sqrt{2^x} \neq x$. If that were the case we would get that $x = \sqrt{37} \approx 6.08$. Checking to see if this is a solution we get $2^{6.08} \approx 67.65 \neq 37$. So how *DO* we "undo" the 2^x? That is where logarithms come in, but we are not going to get to them quite yet. For the time being, note that we could obtain an approximate solution by guessing and checking:

◇ **Example 6.1(f):** Using guessing and checking, find a value of x for which 2^x, is within one one-tenth of 37.

Solution: Since $2^3 = 8$ (I used this because most of us know it) x is probably something like 2^5; checking we find that $2^5 = 32$. This is a bit low but $2^6 = 64$, so we might guess that $x = 5.2$. $2^{5.2} = 36.75$, so we are close. But we were asked to find x to the nearest hundredth, so we need to keep going. Let's keep track of our guesses and the results in a table:

x :	5	6	5.2	5.3	5.22	5.21
2^x :	32	64	36.75	39.40	37.27	37.01

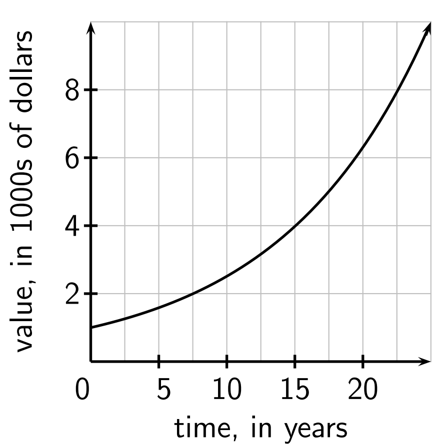
Thus we have that $x \approx 5.21$.

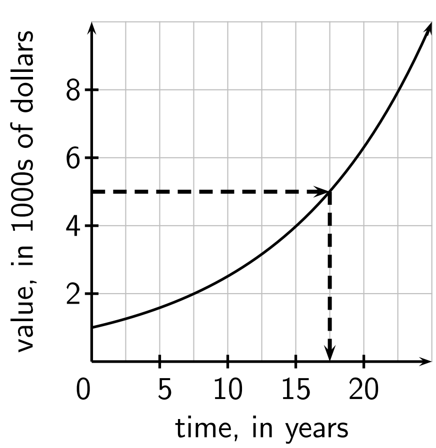
The above process is not very efficient, especially compared with a method you will see soon, but for certain kinds of equations this might be the only way to find a solution. There are also more efficient methods of "guessing and checking" that some of you may see someday.

We can also get approximate "input" values from a graph as well:

◇ **Example 6.1(g):** The graph below and to the left shows the growth of a $1000 investment. Determine how long you would have to wait for the investment to have a value of $5000.

Solution: Looking at the graph below and to the left, we go to $\$5000$ on the value axis and go across horizontally to the graph of the function (see the horizontal dashed line), which is the solid curve. From the point where we meet the curve, we drop straight down to find that the value of our investment will be $\$5000$ of winnings in about 17.5 years. This process is illustrated by the dashed lines on the graph below and to the right.





Section 6.1 Exercises To Solutions

1. Let $f(t) = 150(2)^{-0.3t}$.

   (a) Find $f(8)$ to the nearest hundredth.

   (b) Use guessing and checking to determine when $f(t)$ will equal 50. Continue until you have determined t to the nearest tenth. (This means the value of t to the tenth's place that gives the value closest to 50.)

   (c) Find the average rate of change of f with respect to t from $t = 2$ to $t = 8$.

2. Graph each of the following without using a calculator of any kind. Label the y-intercept and one point on each side of the y-axis with their *exact* coordinates. You may use the facts that $a^{-n} = \frac{1}{a^n}$ and $a^{\frac{1}{n}} = \sqrt[n]{a}$.

   (a) $y = 3^x$ (b) $y = 3^{-x}$ (c) $y = (\frac{1}{5})^x$

3. Graph the function $y = 2^x$, carefully locating the points obtained when $x = 0, 1, 2$. Check your answer with *Desmos*. Then, using what you learned in Section 5.5, graph each of the following, indicating clearly the new points corresponding to the three points on the original graph. If the asymptote $y = 0$ changes, indicate the new asymptote clearly. Check each answer with *Desmos*.

   (a) $y = 2^x + 1$ (b) $y = 2^{x+1}$ (c) $y = 2^{-x}$

   (d) $y = -2^x$ (e) $y = 3(2^x)$ (f) $y = 2^{0.5x}$

4. Enter the function $y = C(2)^{kt}$ in *Desmos*, with sliders for C and k.

   (a) Which of the two parameters affects the y-intercept?

   (b) Which of the two parameters affects the rate of increase of the function more dramatically? What happens when that parameter goes from positive to negative? Give your answer using language from this section.

   (c) On the same grid, sketch the graphs obtained when $C = 1$ and $k = 1, 2, -1$, labelling each with its equation.

   (d) On a different grid from part (c), but on the same grid as each other, sketch the graphs obtained when $k = 1$ and $C = 1, 2, -1$, labelling each with its equation.

5. For $g(x) = 5^x - 3x$, use guessing and checking to approximate the value of x for which $f(x) = 100$. Find your answer to the nearest hundredth, showing all of your guesses and checks in a table.

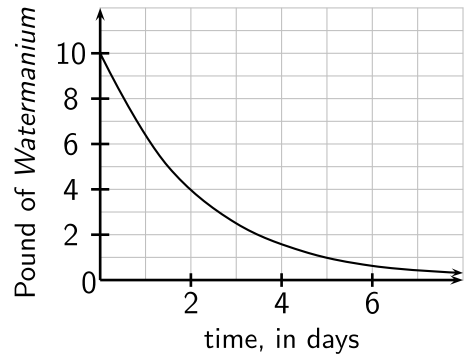
6. The graph below and to the right shows the decay of 10 pounds of *Watermanium* (see Example 6.1(f)). Use it to answer the following.

   (a) How many pounds will be left after two days?

   (b) When will there be two pounds left?

   (c) Find the average rate of change in the amount from zero days to five days.

   (d) Find the average rate of change in the amount from two days to five days.



7. The half-life of ^{210}Bi is 5 days. Suppose that you begin with 80 g.

   (a) Sketch the graph of the amount of ^{210}Bi (on the vertical axis) versus time in days (on the horizontal axis), for twenty days.

   (b) Use your graph to estimate when there would be 30g of ^{210}Bi left. Give your answer as a short, but complete, sentence.

8. The half-life of ^{210}Bi is 5 days. Suppose that you begin with an unknown amount of it, and after 25 days you have 3 grams. How much must you have had to begin with?

9. Money in a certain long-term investment will double every five years. Suppose that you invest $\$500$ in that investment.

   (a) Sketch a graph of the value of your money from time zero to 20 years from the initial investment.

   (b) Use your graph to estimate when you will have $\$3000$.

10. Suppose that you have just won a contest, for which you will be awarded a cash prize. The rules are as follows:

    - This is a one-time cash award. After waiting whatever length of time you wish, you must take your full award all at once. You will receive nothing until that time.
    - You must choose Plan F or Plan G. If you choose Plan F, the amount of cash (in dollars) that you will receive after waiting t years is determined by the function $f(t) = 10(t+1)^2$. If you choose Plan G, your award will be determined by $g(t) = 10(2)^t$.

    Which plan should you choose?

6.2 Exponential Models, The Number e

Performance Criteria:

6. (d) Use exponential models of "real world" situations, and compound interest formulas, to solve problems.

   (e) Solve problems involving the number e.

The following example is a typical application of an exponential function.

◇ **Example 6.2(a):** A culture of bacteria is started. The number of bacteria after t hours is given by $f(t) = 60(3)^{t/2}$.

(a) How many bacteria are there to start with?

(b) How many bacteria will there be after 5 hours?

(c) Use guessing and checking to determine when there will be 500 bacteria. Get the time to enough decimal places to assure that you are within 1 bacteria of 500. **Show each time value that you guess, and the resulting number of bacteria**.

Solution: To answer part (a) we simply let $t = 0$ and find that $f(0) = 60$ bacteria. The answer to (b) is found in the same way: $f(5) = 60(3)^{5/2} = 935$ bacteria. In answering (c) we can already see that t must be less than 5 hours, by our answer to (b). We continue guessing, recording our results in a table:

time, in hours:	3.0	4.0	3.8	3.85	3.86
number of bacteria:	311.7	540.0	483.8	497.3	500.0

The next example will lead us into our next application of exponential functions.

◇ **Example 6.2(b):** Suppose that you invest \$100 in a savings account at 5% interest per year. Your interest is paid at the end of each year, and you put the interest back into the account. Make a table showing how much money you will have in your account at the end of each of the first five years (right after the interest for the year is added to the account). Include the "zeroth year," at the end of which you have your initial \$100.

Solution: The interest for the first year is $0.05 \times \$100 = \$5.00$, so you have $\$100 + \$5.00 = \$105.00$ at the end of the first year. The interest for the second year is $0.05 \times \$105.00 = \$5.25$, and the total at the end of that year is $\$105.00 + \$5.25 = \$110.25$. Continuing this way we get \$115.76, \$121.55 and \$127.63 at the ends of the third, fourth and fifth years, respectively. These results are shown in the table to the right; A stands for "amount," in dollars.

t (yrs)	A (\$)
0	100.00
1	105.00
2	110.25
3	115.76
4	121.55
5	127.63

Let's call the original amount of $100 in the last example the **principal** P, which is "bankerspeak" for the amount of money on which interest is being computed (it could be debt as well as money invested), and let's let r represent the annual interest rate *in decimal form*. The amount of money after one year is then $P + Pr = P(1+r)$. This then becomes the principal for the second year, so the amount after the end of that year is

$$[P(1+r)] + [P(1+r)]r = [P(1+r)](1+r) = P(1+r)^2.$$

Continuing like this, we would find that the amount A after t years would be $A = P(1+r)^t$. This process of increasing the principal by the interest every time it is earned is called **compounding** the interest. In this case the interest is put in at the end of each year, which we called **compounded annually**.

In most cases the compounding is done more often than annually. Let's say it was done monthly instead. You would not earn 5% every month, but one-twelfth of that, so we would replace the r in $(1+r)$ with $\frac{r}{12}$. If we then left the money in for five years it would get compounded twelve times a year for $12 \times 5 = 60$ times. The amount at the end of five years would then be $A = P\left(1+\frac{r}{12}\right)^{12(5)}$. In general we have the following.

Compound Interest Formula

When a principal amount of P is invested for t years at an annual interest rate (in decimal form) of r, compounded n times per year, the total amount A is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

◇ **Example 6.2(c):** Suppose that you invest $1200 for 8 years at 4.5%, compounded semi-annually (twice a year). How much will you have at the end of the 8 years?

Another Example

Solution: The interest rate is $r = 4.5\% = 0.045$, and $n = 2$. We simply substitute these values, along with the time of 8 years, into the formula to get

$$A = 1200\left(1+\frac{0.045}{2}\right)^{2(8)}.$$

You can compute this on your calculator by entering $1200 \times (1 + .045 \div 2) \text{ \^{} } (2 \times 8)$, giving a result of $\$1713.15$.

Here are a couple of comments at this point:

- Try the above calculation yourself to make sure you can get the correct value using your calculator. *Be sure to put parentheses around the* 2×8. You should learn to enter it all at once, rather than making intermediate computations and rounding, which will cause errors in your answer.

- *The power of compounding is significant!* If your interest was just calculated at the end of the eight years (but giving you 4.5% per year), your amount would be $A = 1200 + 1200(.045)(8) = \$1632.00$. The difference is about \$80, which is significant for the size of the investment. Interest computed without compounding is called **simple interest**.

The following example shows that although we are better off having our money compounded as often as possible, there is a limit to how much interest we can earn as the compounding is done more and more often.

◇ **Example 6.2(d):** Suppose that you invest \$1000 at 5% annual interest rate for two years. Find the amounts you would have at the end of the year if the interest was compounded quarterly, monthly, weekly, daily and hourly. (Assume there are 52 weeks in a year, and 365.25 days in a year.)

Solution: Using the formula $A = 1000\left(1 + \dfrac{0.05}{n}\right)^{2n}$ for $n = 4, 12, 52, 365.25$ and 8766 we get \$1104.49, \$1104.94, \$1105.12, \$1105.16 and \$1105.17.

What we would find as we compounded even more often than hourly is that the amount would continue to increase, but by less and less. Rounded, the amounts for compounding more often would all round to \$1105.17. Using language and notation from the last chapter, we could say

As $n \to \infty$, $A \to \$1105.17$

Let's see (sort of) why that happens. Using the property of exponents that $x^{ab} = (x^a)^b$, and remembering that the value 0.05 is our interest rate r, the equation for the amount A can be rewritten as

$$A = 1000\left(1 + \frac{0.05}{n}\right)^{2n} = 1000\left[\left(1 + \frac{0.05}{n}\right)^{n}\right]^2 = 1000\left[\left(1 + \frac{r}{n}\right)^{n}\right]^2$$

It turns out that as $n \to \infty$, the quantity $(1 + \frac{r}{n})^n$ goes to the value $(2.718...)^r$, where 2.718... is a decimal number that never repeats or ends. That number is one of the four most important numbers in math, along with 0, 1 and π, and it gets its own symbol, e. The final upshot of all this is that

$$\text{as } n \to \infty,\ A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n}\right]^t \to P(e^r)^t = Pe^{rt}.$$

The last expression is the amount one would have if the interest was **compounded continuously**, which is a little hard to grasp, but sort of means that the interest is compounded "every instant" and added in to the principal as "instantaneously." This is summarized as follows.

Continuous Compound Interest Formula

When P dollars are invested for t years at an annual interest rate of r, compounded continuously, the amount A of money that results is given by

$$A = Pe^{rt}$$

◇ **Example 6.2(e):** If you invest \$500 at 3.5% compounded continuously for ten years, how much will you have at the end of that time?

Solution: The trick here is not the setup, but understanding how to use your calculator. The amount is given by $A = 500e^{0.035(10)}$. All calculators allow us to take e to a power (which is all we ever do with e); this operation is found above the LN button. To do this computation with a graphing calculator you will type in $500 \times$ 2nd LN ($.035 \times 10$) and hit ENTER. With other calculators, you might do $.035 \times 10 =$ 2nd LN $\times 500$. The result is \$709.53.

In an earlier example we used the exponential model $f(t) = 60(3)^{t/2}$, which can be rewritten as $f(t) = 60(3)^{0.5t} = 60(3^{0.5})^t$. It turns out that $3^{0.5} = e^{0.549...}$ (you can verify this with your calculator), so we could then write $f(t) = 60(e^{0.549...})^t = 60e^{0.549...t}$. This sort of thing can always be done, so we often use e as the base for our exponential models.

Exponential Growth and Decay Model

When a quantity starts with an amount A_0 or a number N_0 and grows or decays exponentially, the amount A or number N after time t is given by

$$A = A_0e^{kt} \qquad \text{or} \qquad N = N_0e^{kt}$$

for some constant k, called the **growth rate** or **decay rate**. When k is a growth rate it is positive, and when k is a decay rate it is negative.

Section 6.2 Exercises

To Solutions

1. (a) You invest \$500 at an annual interest rate of 4 %. If it is compounded monthly, how much will you have after 3 years?

   (b) If you invest \$2500 at an annual interest rate of 1.25%, compounded weekly, how much will you have after 7 years?

   (c) You invest \$10,000 at an annual interest rate of 6.25%, compounded semi-annually (twice a year). How much will you have after 30 years?

   (d) For the scenario of part (c), what is your average rate of increase over the 30 years?

2. You are going to invest some money at an annual interest rate of 2.5%, compounded quarterly. In six years you want to have \$1000. How much do you need to invest? **Note that here you are given A, not P.**

3. How much will be in an account after \$2500 is invested at 8% for 4 years, compounded continuously?

4. (a) How much money will result from continuous compounding if \$3000 is invested for 9 years at $5\frac{1}{4}\%$ annual interest?

   (b) What is the average rate of increase over those nine years?

5. How much would have to be invested at 12%, compounded continuously, in order to have \$50,000 in 18 years?

6. You are going to invest \$500 for 10 years, and it will be compounded continuously. What interest rate would you have to be getting in order to have \$1000 at the end of those 10 years? **You will have to use guessing and checking to answer this. Find your answer to the nearest tenth of a percent.**

7. In 1985 the growth rate of the population of India was 2.2% per year, and the population was 762 million. Assuming continuous exponential growth, the number of people t years after 1985 is given by the equation $N(t) = 762e^{0.022t}$, where N **is in millions** of people.

   (a) What will the population of India be in 2010? Round to the nearest million.

   (b) Based on this model, what is the average rate of change in the population, with respect to time, from 1985 to 2010?

8. The height h (in feet) of a certain kind of tree that is t years old is given by

$$h = \frac{120}{1 + 200e^{-0.2t}} \ .$$

   (a) Use this to find the height of a tree that is 30 years old.

   (b) This type of equation is called a *logistic equation*. It models the growth of things that start growing slowly, then experience a period of rapid growth, then level off. Sketch the graph of the function for the first 60 years of growth of a tree. Use your graphing calculator, or find values of h for $t = 0, 10, 20, ..., 60$.

   (c) Around what age does the tree seem to be growing most rapidly?

9. For the variety of tree from the previous exercise,

   (a) What is the average rate of growth for the first 30 years?

   (b) What is the average rate of growth from 20 years to 30 years?

10. The expression e^u is greater than zero, no matter what u is. (This is actually saying the range of $y = a^x$ is $(0, \infty)$ for all $a > 0$, $a \neq 1$, as noted in Section 6.1.) This is a very important fact - use it to do the following.

   (a) Find all values of x for which $f(x) = 0$, where $f(x) = x^2e^x - 9e^x$. How do you know that you have given ALL such values?

   (b) Find all values of x for which $f(x) = 0$, where $f(x) = x^2e^{-x} - 9xe^{-x}$.

6.3 Logarithms

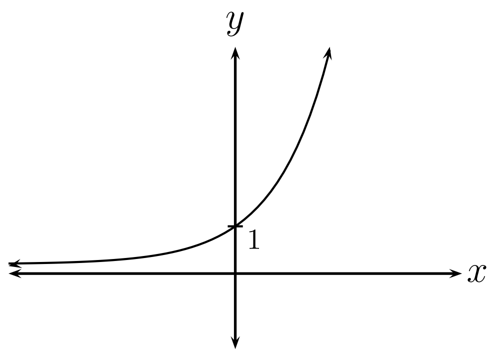
Performance Criteria:

6. (f) Evaluate logarithms using the definition.

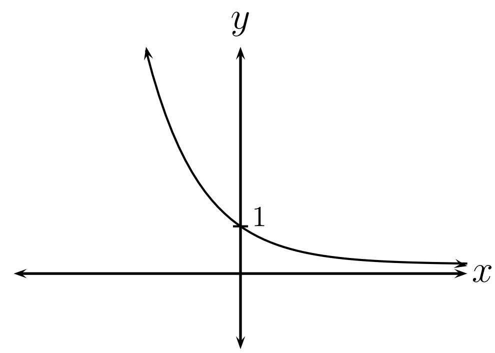
   (g) Change an exponential equation into logarithmic form and vice versa; use this to solve equations containing logarithms.

The Definition of a Logarithm Function

Recall the graphs of $y = a^x$ for $a > 1$ and $0 < a < 1$:

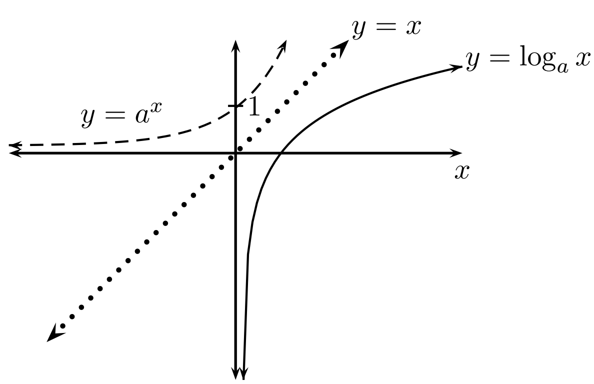


$f(x) = a^x,\ (a > 1)$

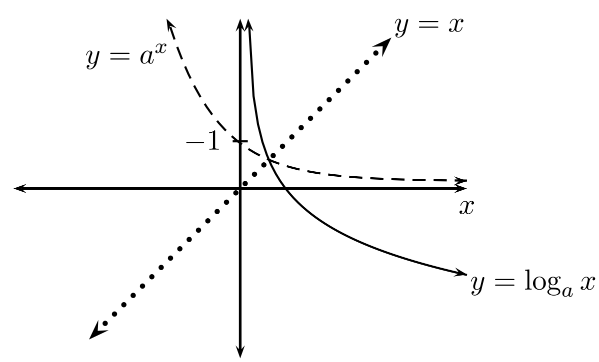


$f(x) = a^x,\ (0 < a < 1)$

In both cases we can see that the function is one-to-one, with range $y > 0$. Therefore an inverse function exists, with domain $x > 0$. We call the inverse of $y = a^x$ the **logarithm base** a, denoted as $y = \log_a x$. Remembering that the graph of the inverse of a function is its reflection across the line $y = x$, we can see that the graph of $y = \log_a x$ can have the two appearances below, depending on whether a is greater than or less than one.



$y = \log_a x,\ (a > 1)$



$y = \log_a x,\ (0 < a < 1)$

Remember that square root is the inverse function for "squared," at least as long as the values being considered are greater than or equal to zero. Based on this, the square root of a number x is the (non-negative) value that can be squared to get x. Similarly, the logarithm base a of $x > 0$ (because the domain of the logarithm is $(0, \infty)$) is the exponent to which a can be taken to get x. Clear as mud? Let's look at two examples:

◇ **Example 6.3(a):** Find $\log_2 8$ and $\log_3 \frac{1}{9}$. Another Example

Solution: $\log_2 8$ the the power that 2 can be taken to get 8. Since $2^3 = 8$, $\log_2 8 = 3$. For the second one, we are looking for a power of 3 that equals $\frac{1}{9}$. Because $3^2 = 9$, $3^{-2} = \frac{1}{9}$ and we have $\log_3 \frac{1}{9} = -2$.

Note that what we were essentially doing in this example was solving the two equations $2^x = 8$ and $3^x = \frac{1}{9}$, which shows that every question about a logarithm can always be restated as a question about an exponent. All of this is summarized in the following definition.

> **Definition of the Logarithm Base a, Version 1**
>
> Suppose that $a > 0$ and $a \neq 1$. For any $x > 0$, $\log_a x$ is the power that a must be raised to in order to get x. Another way of saying this is that
>
> $y = \log_a x$ is equivalent to $a^y = x$

Here x and y are variables, but a is a constant, the **base** of the logarithm. There are infinitely many logarithm functions, since there are infinitely many values of a that we can use. (Remember, though, that a must be positive and not equal to one.) Note that the domain of any logarithm function is $(0, \infty)$ because that set is the range of any exponential function.

The easiest way to apply the above when working with an equation with a logarithm in it is to get it into the form $\log_a B = C$ and then use a "circular motion" to transform it into an exponential equation, as I've attempted to show below:

$$\log_a B = C \quad \Longrightarrow \quad a^C = B \tag{1}$$

"a to the C equals B"

◇ **Example 6.3(b):** Change $3 = \log_5 125 = 3$ and $\log_2 R = x^4$ to exponential form.

Solution: To work with $3 = \log_5 125 = 3$ we rewrite it as $\log_5 125 = 3$ and apply (1) above to get $5^3 = 125$. Doing the same with the second equation results in $2^{x^4} = R$.

◇ **Example 6.3(c):** Solve the equation $\log_3 4x + 5 = 7$.

Solution: In order to apply (1) we must first get the equation in the form $\log_3 A = B$, which we do by subtracting five from both sides.

$$\begin{aligned} \log_3 4x + 5 &= 7 \\ \log_3 4x &= 2 \\ 3^2 &= 4x \\ 9 &= 4x \\ x &= \tfrac{9}{4} \end{aligned}$$

Note that if the equation from the previous example woud have been instead $\log_3(4x+5) = 7$ we could have applied (1) right away.

◇ **Example 6.3(d):** Solve the equation $\log_3(4x+5) = 7$.

Solution:

$$\begin{aligned} \log_3(4x+5) &= 7 \\ 3^7 &= 4x+5 \\ 2187 &= 4x+5 \\ 2182 &= 4x \\ x &= \tfrac{2182}{4} \end{aligned}$$

◇ **Example 6.3(e):** Solve the equation $\log_4 \frac{1}{64} = 2x$.

Solution: Changing the equation into exponential form results in $4^{2x} = \frac{1}{64}$. Noting that $\frac{1}{64} = 4^{-3}$, we must have $2x = -3$ and $x = -\frac{3}{2}$.

Usually we will wish to change from logarithm form to exponential form, as in the last example, but we will occasionally want to go the other way.

◇ **Example 6.3(f):** Change $4^{x-1} = 256$ to logarithm form. Another Example

Solution: In this case, from the definition of a logarithm we have $a = 4$, $y = x-1$ and $x = 256$. Putting these into $y = \log_a x$ results in $x - 1 = \log_4 256$.

Logarithms With a Calculator

In general, we can't find logarithms "by hand" - we need to enlist the help of our calculator. Before discussing how to do this, recall that there is a logarithm function *for every positive number except* 1, and your calculator is not going to be able to have a key for every one of them! There are two logarithms that are used most often:

- $\log_{10}$, "log base 10". This is called the **common logarithm**, and we don't generally write the base with it. It is the key on your calculator just labeled LOG.
- $\log_e$, "log base e". This is called the **natural logarithm**, and we write "ln" for it, with no base indicated. It is the key on your calculator labeled LN.

◇ **Example 6.3(g):** Find $\log 23.4$ and $\ln 126$ to the nearest thousandth.

Solution: $\log 23.4 = 1.369$ and $\ln 126 = 4.836$.

You will find that we can do just about everything we ever want to with just those two logarithms, but should you ever wish to find the logarithm of a number with some base other than 10 or e there are a couple options. Some calculators will compute logarithms with other bases, or you can find sites online that will do this. The other option is to use the following, the derivation of which we'll see later.

Change of Base Formula

$$\log_a B = \frac{\log B}{\log a}$$

◇ **Example 6.3(h):** Find $\log_3 17.2$ to the nearest hundredth.

Solution: Applying the formula we get $\log_3 17.2 = \dfrac{\log 17.2}{\log 3} = 2.59.$

Logarithms as Inverses of Exponential Functions

Use your calculator to find $\log 10^{3.49}$. You should get 3.49, because the common logarithm "undoes" 10^x and vice versa. That is, the inverse of $f(x) = 10^x$ is $f^{-1}(x) = \log x$. Similarly, the inverse of $g(x) = e^x$ is $g^{-1}(x) = \ln x$. This is the true power of logarithms, which can be summarized for all logarithms as follows:

Inverse Property of Logarithms, Part I

$\log_a(a^u) = u$ and $a^{\log_a u} = u$

Notice that 10^x and LOG share the same key on your calculator, as do e^x and LN. This is, of course, no coincidence!

◇ **Example 6.3(i):** Find $\log_5 5^\pi$ and $\log_7 4^{1.2}$.

Solution: $\log_5 5^\pi = \pi$. We cannot do the same for $\log_7 4^{1.2}$ because the base of the logarithm does not match the base of the exponent. We would need to use the change of base formula to find its value; we'll skip that, since that was not the point of this example.

There is another formulation of the definition of logarithms that we can use to find inverses of functions that contain exponential or logarithm functions:

Definition of a Logarithm Base a, Version 2

The inverse of $y = \log_a x$ is $y = a^x$, and vice-versa.

◇ **Example 6.3(j):** Find the inverse of $f(x) = \log_2 3x + 5$.

Solution: As we did in Chapter 5, we determine the operations, and the order in which they are performed, for f. x is first multiplied by 3, then $\log_2$ is performed on that result and, finally, five is added. For the inverse we must first subtract five, and we take three to the resulting power. Finally, that result is divided by two or, equivalently, multiplied by $\frac{1}{2}$:

$$f^{-1}(x) = \frac{3^{x-5}}{2} = \tfrac{1}{2}(3)^{x-5}.$$

◇ **Example 6.3(j):** Find the inverse of $g(x) = 5^{x+1}$.

Solution: The function adds one, then raises five to the resulting power. For the inverse we must first take the logarithm base five, then subtract one, so

$$g^{-1}(x) = \log_5 x - 1.$$

Section 6.3 Exercises

To Solutions

1. Find any of the following that are possible.

   (a) $\log_4 1$ (b) $\log_3 \frac{1}{81}$ (c) $\log_4(-16)$ (d) $\log_9 3$

2. Change each of the following equations containing logarithms to their equivalent exponential equations.

   (a) $\log_2 \frac{1}{8} = -3$ (b) $\log_{10} 10,000 = 4$ (c) $\log_9 3 = \frac{1}{2}$

3. Change each of the following exponential equations into equivalent logarithm equations.

   (a) $4^3 = 64$ (b) $10^{-2} = \frac{1}{100}$ (c) $8^{2/3} = 4$

4. Use the definition of a logarithm to find **two** ordered pairs that satisfy $y = \log_5 x$.

5. Use the definition of a logarithm to solve (a) $\log_4 x = \frac{3}{2}$ (b) $\log_9 x = -2$

6. Change $7^x = 100p$ to logarithmic form.

7. Find $\log_4 \frac{1}{16}$.

8. Use your calculator to find each of the following. Round to the nearest hundredth.

(a) $\log 12$ (b) $\ln 17.4$ (c) $\ln 0.035$ (d) $\log 4.6$

9. Find each of the following.

(a) $\log(10^{6.2})$ (b) $10^{\log 0.89}$ (c) $e^{\ln 4}$ (d) $\ln(e^{7.1})$

10. Find each of the following.

(a) $\log_3 3^{7x}$ (b) $\ln e^{x^2}$ (c) $10^{\log 12.3}$ (d) $7^{\log_7(x-5)}$

11. Find each, rounded to three significant digits: (a) $\log_3 4.9$ (b) $\log_8 0.35$

12. Read Example 6.3(h), then compute $\dfrac{\ln 17.2}{\ln 3}$. How does the result compare to what was obtained in the example?

13. Solve each logarithm equation for the given unknown.

(a) $\log_2 3x = 1,\ \ x$

(b) $2 = \log_3(w-4),\ \ w$

(c) $\log_{10}\left(\dfrac{7}{t}\right) = 0,\ \ t$

(d) $\ln(2x-1) = 3,\ \ x$

(e) $d = \log_5(ax) + b,\ \ x$

(f) $\log_5(ax+b) = d,\ \ x$

(g) $\log_5(ax+b) = d,\ \ b$

14. Give the inverse of each function.

(a) $f(x) = \ln(3x+1)$

(b) $y = 2^x + 5$

(c) $g(x) = 2(3)^x$

(d) $h(x) = \dfrac{\log_2(x+1)}{7}$

(e) $y = \log(3x-4)$

(f) $f(x) = 3^{2x+1}$

6.4 Some Applications of Logarithms

Performance Criteria:

6. (h) Use the inverse property of the natural logarithm and common logarithm to solve problems.

In this section we will see some applications of logarithms; a large part of our interest in logarithms is that they will allow us to solve exponential equations for an unknown that is in the exponent of the equation without guessing and checking, as we did earlier. The key is the inverse property of logarithms, adapted specifically to the natural logarithm:

$$\log_a(a^u) = u \text{ so, in particular, } \ln e^u = u$$

◇ **Example 6.4(a):** Solve $1.7 = 5.8e^{-0.2t}$ for t, to the nearest hundredth.

1.7	$=$	$5.8e^{-0.2t}$	the original equation
$\frac{1.7}{5.8}$	$=$	$e^{-0.2t}$	divide both sides by 5.8
$\ln\left(\frac{1.7}{5.8}\right)$	$=$	$\ln e^{-0.2t}$	take the natural logarithm of both sides
$\ln\left(\frac{1.7}{5.8}\right)$	$=$	$-0.2t$	use the fact that $\ln e^u = u$
$\dfrac{\ln\left(\frac{1.7}{5.8}\right)}{-0.2}$	$=$	t	divide both sides by -0.2
6.14	$=$	t	compute the answer and round as directed

You should try this last computation on your calculator, to make sure that you know how to do it.

◇ **Example 6.4(b):** The growth of a culture of bacteria can be modeled by the equation $N = N_0e^{kt}$, where N is the number of bacteria at time t days, N_0 is the number of bacteria at time zero, and k is the growth rate. If the doubling time of the bacteria is 4 days, what is the growth rate, to two significant digits?

Solution: Although we don't know the actual number of bacteria at any time, we DO know that when $t = 4$ days the number of bacteria will be twice the initial amount, or $2N_0$. We can substitute this into the equation along with $t = 4$ and solve for k:

$$\begin{aligned}
2N_0 &= N_0 e^{k(4)} && \text{the original equation}\\
2 &= e^{4k} && \text{divide both sides by } N_0\\
\ln 2 &= \ln e^{4k} && \text{take the natural logarithm of both sides}\\
\ln 2 &= 4k && \text{use the fact that } \ln e^u = u\\
\frac{\ln 2}{4} &= k && \text{divide both sides by } 4\\
0.17 &= k && \text{compute the answer and round as directed}
\end{aligned}$$

There are many, many applications of exponential functions in science and engineering; although perhaps not as common, logarithms functions have a number of applications as well. One that some of you may have seen already is the idea of pH, used to measure the acidity or basicity of a a chemical substance, in liquid form. The pH of a liquid is defined to be

$$\text{pH} = -\log[\text{H}^+],$$

where H^+ is the hydrogen ion concentration in moles per liter. Another application is in the study of sound intensity, which is measured in something called decibels. Sound intensity is really just a pressure, as are waves that travel through the earth after an earthquake, so the Richter scale for describing the intensity of an earthquake is also based on logarithms.

Let's consider a specific example. *Ultrasound imaging* is a method of looking inside a person's body by sending sound waves into the body, where they bounce off of tissue boundaries and return to the surface to be recorded. The word *ultrasound* means that the sound is at frequencies higher than we can hear. The sound is sent and received by a device called a *transducer*. As the sound waves travel through tissues they lose energy, a process called *attenuation*. This loss of energy is expressed in units of **decibels**, which is abbreviated as dB. We have the following equation:

$$\text{attenuation} = 10\log\frac{I_2}{I_1} \tag{1}$$

where I_1 and I_2 are the intensities at two points 1 and 2, with point 2 being beyond point 1 on the path of travel of the ultrasound signal. The units of intensity are in units of power per unit of area (commonly watts per square centimeter).

⋄ **Example 6.4(c):** The intensity of a signal from a 12 megahertz (MHz) transducer at a depth of 2 cm under the skin is one-fourth of what it is at the surface. What is the attenuation at that depth, to the hundredth's place?

Solution: Here I_1 is the intensity at the surface of the skin and I_2 is the intensity at 2 centimeters under the surface. We are given that $I_2 = \frac{1}{4}I_1$, so

$$\text{attenuation} = 10\log\frac{I_2}{I_1} = 10\log\frac{\frac{1}{4}I_1}{I_1} = 10\log\frac{1}{4} = -6.02 \text{ dB}$$

The attenuation is -6.02 decibels.

Attenuation comes up in many other settings that involve transmission of signals through some medium like air, wire, fiber optics, etc.

We can use the inverse property of logarithms to solve equations that begin as logarithm equations as well, as illustrated in the next example.

◇ **Example 6.4(d):** A 12MHz (megahertz) ultrasound transducer is applied to the skin, and at a depth of 5.5 cm the intensity is $8.92 \times 10^{-3} \mu\text{W}/\text{cm}^2$ (micro-watts per square centimeter). If this is an attenuation of -34 decibels, what is the intensity of the signal at the transducer, on the skin? Round to three significant figures.

Solution: Here $I_2 = 8.92 \times 10^{-3}$, attenuation is -34, and we are trying to determine I_1. The answer is obtained by substituting the known values into equation (1) and solving:

$$\begin{aligned}
-34 &= 10\log\frac{8.92\times 10^{-3}}{I_1} && \text{equation (1) with values substituted}\\
-3.4 &= \log\frac{8.92\times 10^{-3}}{I_1} && \text{divide both sides by } 10\\
10^{-3.4} &= \frac{8.92\times 10^{-3}}{I_1} && \text{apply the definition of a logarithm}\\
10^{-3.4}I_1 &= 8.92\times 10^{-3} && \text{multiply both sides by } I_1\\
I_1 &= \frac{8.92\times 10^{-3}}{10^{-3.4}} && \text{divide both sides by } 10^{-3.4}\\
I_1 &= 22.4 && \text{compute the answer and round as directed}
\end{aligned}$$

The intensity at the transducer is $22.4\mu\text{W}/\text{cm}^2$.

Section 6.4 Exercises To Solutions

1. Solve $280 = 4e^{2.3k}$, rounding your answer to the nearest hundredth.

2. Solve $15 = 3000e^{-.04t}$, rounding your answer to two significant figures.

3. You are going to invest \$3000 at 3.5% interest, compounded continuously. How long do you need to invest it in order to have \$5000?

4. The growth of a population of rabbits in a ten square mile plot is modeled by $N = 1300e^{0.23t}$, with t in years.

   (a) When will the population of rabbits reach 10,000? Round to the nearest tenth of a year.

   (b) What is the doubling time for the population, to the nearest tenth of a year?

5. Recall the equation $N(t) = 762e^{0.022t}$ for the number of people in India t years after 1985, where N **is in millions** of people.

   (a) According to this model, when should the population of India will reach 2 billion?

   (b) Compute what the current population of India should be, based on this model. Then look up the population of India. Has the rate of population growth increased, or decreased from what it was when this model was developed?

6. Previously we saw that the height h (in feet) of a certain kind of tree that is t years old is given by
$$h = \frac{120}{1 + 200e^{-0.2t}} .$$
When, to the nearest whole year, will such a tree reach a height of 100 feet?

7. The intensity of a particular 8MHz ultrasound transducer at the point where it is applied to the skin is $20\mu\mathrm{W/cm^2}$. The attenuation for such a transducer is -4dB per centimeter of depth into soft tissue. Based on equation (1) (the equation for attenuation), what intensity will a signal from the transducer have at a depth of 3 centimeters? Give your answer to two significant figures; as usual, include units!

8. Consider again the transducer from Exercise 6. If we let d represent depth below the skin, in centimeters, then the attenuation at depth d is $-4d$. From the attenuation formula we then get
$$-4d = 10 \log \frac{I_2}{20}$$
as an equation relating the intensity and depth. Solve this equation for I_2, to get I_2 as a function of d.

9. The depth at which the intensity of an ultrasound signal has decreased to half the intensity at the skin is called the **half-intensity depth**. What is the attenuation at that depth? Round to the nearest tenth of a decibel. (**Hint:** I_2 is what fraction of I_1 in this case?)

10. The growth model obtained for the bacteria in Example 6.4(b) is $N = N_0e^{0.17t}$, Using this model, when will the population of bacteria be triple what it was to begin?

11. $3000 is invested at an annual rate of $5\frac{1}{4}$%, compounded continuously. How long will it take to have $5000?

12. The half-life of a medication is three hours. Find the rate of decay, k.

13. Assume that the number of mice in a barn is modelled by an equation of the form $N = N_0e^{kt}$.

   (a) At the time that is being considered to be time zero there are estimated to be 500 mice in the barn. This tells us that when _____ $= 0$, _____ $= 500$.

(b) Use your answers to (a), along with the equation $N = N_0e^{kt}$ to find N_0. It is a constant, so once its value is determined it remains the same, so you can put that value into the equation. Write down what you know the equation looks like at this point.

(c) Two months later it is estimated that there are 590 mice in the barn. Use this information and your equation from (b) to determine the value of k, to the thousandth's place. k is also a constant, so once you know its value you can put into the equation as well - write down the result.

(d) Now that you know N_0 and k, determine how many mice we'd expect to be in the barn a year after time zero.

14. A person takes 150mg of a medication, and 4 hours later there are 120mg in their system. Assuming that the amount A at any time t is given by $A = A_0e^{kt}$, use a method like that of the previous exercise to determine how many milligrams will be in the person's body after 10 hours, to the nearest milligram. (Again find k to the thousandth's place.)

6.5 Properties of Logarithms

Performance Criteria:

6. (i) Use properties of logarithms to write a single logarithm as a linear combination of logarithms, or to write a linear combination of logarithms as a single logarithm.

   (j) Apply properties of logarithms to solve logarithm equations.

In the last section we saw how to use the inverse property of logarithms to solve equations like $1.7 = 5.8e^{-0.2t}$. In certain applications like compound interest we will instead wish to solve an equation like $5000 = 1000(1.0125)^{4t}$, and we will not be able to use the inverse property of the logarithm directly because our calculators do not have a $\log_{1.0125}$. In this section we will see some algebraic properties of logarithms, one of which will allow us to solve such problems. We begin with the following.

◇ **Example 6.5(a):** Determine whether, in general, $\log_a(u+v) = \log_a u + \log_a v$.

Solution: Let's test this equation for $\log_{10}$, with $u = 31$ and $v = 7$:

$$\log(u+v) = \log 38 = 1.58 \quad \text{and} \quad \log u + \log v = \log 31 + \log 7 = 1.49 + 0.85 = 2.34$$

This shows that $\log_a(u+v) \neq \log_a u + \log_a v$.

This should not be surprising. About the only thing we can do to a sum that is equivalent to doing it to each item in the sum and *THEN* adding is multiplication:

$$z(x+y) = zx + zy, \quad \text{but} \quad (x+y)^2 \neq x^2 + y^2, \quad \sqrt{x+y} \neq \sqrt{x} + \sqrt{y}, \quad a^{x+y} \neq a^x + a^y.$$

When you go into trigonometry and use a function called "cos" (that is, the *cosine* function) $\cos(x+y) \neq \cos x + \cos y$. All is not lost however! Even though $a^{x+y} \neq a^x + a^y$, we *DO* have the property that $a^{x+y} = a^x a^y$, and a similar thing happens with logarithms.

◇ **Example 6.5(b):** Determine which of $\log_a(u+v)$, $\log_a(u-v)$, $\log_a(uv)$ and $\log_a\left(\frac{u}{v}\right)$ seem to relate to $\log_a u$ and $\log_a v$, and how.

Solution: Let's again let $a = 10$, $u = 31$ and $v = 7$, and compute the values of all the above expressions, rounding to the nearest hundredth:

$$\log u = 1.49, \qquad \log v = 0.85, \qquad \log(u+v) = 1.58,$$

$$\log(u-v) = 1.38, \qquad \log(uv) = 2.34, \qquad \log\left(\frac{u}{v}\right) = 0.65$$

Note that $\log u + \log v = 2.34$ and $\log u - \log v = 0.64$, which are (allowing for rounding) equal to the values of $\log(uv)$ and $\log(\frac{u}{v})$, respectively. From this, it appears that

$$\log_a uv = \log_a u + \log_a v \qquad \text{and} \qquad \log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$$

The results of this example were not a fluke due to the numbers we chose. There are three algebraic properties of logarithms we'll use, two of which you just saw. Including the third, they can be summarized as follows.

Properties of Logarithms

For $a > 0$ and $a \neq 1$, and for $u > 0$ and $v > 0$,

- $\log_a uv = \log_a u + \log_a v$
- $\log_a \left(\frac{u}{v}\right) = \log_a u - \log_a v$
- $\log_a u^r = r \log_a u$

In this section we'll see some ways these properties can be used to manipulate logarithm expressions, and we'll use the first two along with the definition of a logarithm to solve equations containing logarithms. In the next section we'll see that the third property is an especially powerful tool for solving exponential equations.

◇ **Example 6.5(c):** Apply the properties of logarithms to write $\log_4 \frac{\sqrt{x}}{y^3}$ in terms of logarithms of x and y.

Solution: First we note that $\sqrt{x} = x^{\frac{1}{2}}$. It should be clear that we will apply the second and third properties, but the order may not be so clear. We apply the second property first, and there are two ways to see why this is:

1) Since the numerator and denominator have different exponents, we don't have something of the form $\log_a u^r$. Instead we have $\log_a \left(\frac{u}{v}\right)$, where $u = x^{\frac{1}{2}}$ and $v = y^3$.
2) If we knew values for x and y and wanted to compute the logarithm, we would apply the exponents first, then divide, then perform the logarithm. In the same manner that we invert functions, we have to undo the division first here, then undo the exponents.

So, undoing the division with the second property, then undoing the exponents with the third, we get

$$\log_4 \frac{\sqrt{x}}{y^3} = \log_4 \frac{x^{\frac{1}{2}}}{y^3} = \log_4 x^{\frac{1}{2}} - \log_4 y^3 = \frac{1}{2}\log_4 x - 3\log_4 y$$

◇ **Example 6.5(d):** Apply the properties of logarithms to write $5\ln x + 2\ln(x+3)$ as a single logarithm. Another Example

Solution: You might recognize that what we are doing here is reversing the procedure from the previous exercise:

$$5\ln x + 2\ln(x+3) = \ln x^5 + \ln(x+3)^2 = \ln x^5(x+3)^2$$

This second example gives us a hint as to how we solve an equation like

$$\log_2 x + \log_2(x+2) = 3 \ .$$

The procedure goes like this:

- Use the above properties to combine logarithms and get just one logarithm.
- Apply the definition of a logarithm to get an exponential equation.
- Evaluate the exponent and solve the resulting equation.
- Check to see that the logarithms in the original equation are defined for the value(s) of x that you found.

Note that you can't simply discard a value of x because it is negative; you need to see if it causes a logarithm to be evaluated for a negative number.

◇ **Example 6.5(e):** Solve the equation $\log_2 x + \log_2(x+2) = 3$. Another Example

Solution: Applying the above procedure we get

$\log_2 x + \log_2(x+2)$	$=$	3	the original equation
$\log_2 x(x+2)$	$=$	3	apply the first property to combine the two logarithms
$x(x+2)$	$=$	2^3	apply the definition of a logarithm
x^2+2x	$=$	8	distribute and square
x^2+2x-8	$=$	0	continue solving...
$(x+4)(x-2)$	$=$	0	

Here we have the solutions $x=-4,\ 2,$ but if $x=-4$ both $\log_2 x$ and $\log_2(x+2)$ are undefined, so the only solution is $x=2$.

◇ **Example 6.5(f):** Solve $\ln(8-x) = \ln(3x-2) + \ln(x+2)$. Another Example

Another Example

Solution: There are two ways to solve this problem. First we'll do it like the one above, which requires that we get all terms on one side and apply the properties of logarithms to turn them all into one logarithm:

$$\begin{aligned}
\ln(8-x) &= \ln(3x-2)+\ln(x+2)\\
0 &= \ln(3x-2)+\ln(x+2)-\ln(8-x)\\
0 &= \ln\frac{(3x-2)(x+2)}{(8-x)}\\
e^0 &= \frac{(3x-2)(x+2)}{(8-x)}
\end{aligned}$$

$$\begin{aligned} 1 &= \frac{(3x-2)(x+2)}{(8-x)} \\ 8-x &= 3x^2+4x-4 \\ 0 &= 3x^2+5x-12 \\ 0 &= (3x-4)(x+3) \end{aligned}$$

Here we have the solutions $x = -3, \frac{4}{3}$, but if $x = -3$ both $\ln(3x-2)$ and $\ln(x+2)$ are undefined, so the only solution is $x = \frac{4}{3}$.

The other way to solve this problem is to combine the two terms on the right side and note that if $\ln A = \ln B$, then $A = B$ because logarithms are one-to-one:

$$\begin{aligned} \ln(8-x) &= \ln(3x-2)+\ln(x+2) \\ \ln(8-x) &= \ln(3x-2)(x+2) \\ (8-x) &= (3x-2)(x+2) \\ 8-x &= 3x^2+4x-4 \end{aligned}$$

The remaining steps are the same as for the other method.

Section 6.5 Exercises To Solutions

1. Apply the above properties to write each of the following logarithms in terms of logarithms of x and y.

   (a) $\log_4 \frac{x}{y^2}$
   (b) $\log_5 x^3\sqrt{y}$
   (c) $\ln \sqrt[3]{x^2y}$
   (d) $\ln \frac{\sqrt{x}}{y^3}$

2. Apply the above properties to write each of the following expressions as one logarithm, **without negative or fractional exponents**.

   (a) $2\log x + \log y$
   (b) $\frac{1}{2}\log_2 x - 3\log_2 y$
   (c) $\ln x - \ln y - \ln z$
   (d) $\log_8 y - \frac{3}{2}\log_8 z$
   (e) $\ln(x+1) - \ln(x^2-1)$
   (f) $2\log x + 4\log x$

3. Solve each of the following logarithm equations.

   (a) $\log_3 x + \log_3 9 = 1$
   (b) $\log_6 x + \log_6(x-1) = 1$
   (c) $\log_4(x-2) - \log_4(x+1) = 1$
   (d) $\log_8 x = \frac{2}{3} - \log_8(x-3)$
   (e) $\log_2 x + \log_2(x-2) = 3$
   (f) $\log 100 - \log x = 3$

4. Solve $\log_2(x-3)+\log_2(x-1)=3$ for x.

5. Write $\log_3\left(\frac{x}{y^5}\right)$ in terms of $\log_3 x$ and $\log_3 y$.

6. Write $\frac{1}{2}\log x+3\log y$ as one logarithm, without negative or fractional or negative exponents.

7. Solve $\log_2(x+7)+\log_2 x=3$.

8. Write $\log\left(\frac{\sqrt{x}}{yz^3}\right)$ in terms of $\log x$, $\log y$ and $\log z$.

6.6 More Applications of Exponential and Logarithm Functions

Performance Criteria:

6. (k) Solve an applied problem using a given exponential or logarithmic model.

   (l) Create an exponential model for a given situation.

Solving General Exponential Equations

Suppose that you were to invest $1000 at an annual interest rate of 5%, compounded quarterly. If you wanted to find out how many years it would take for the value of your investment to reach $5000, you would need to solve the equation $5000 = 1000(1.0125)^{4t}$. The method used is similar to what we did in Example 6.4(a) to solve $1.7 = 5.8e^{-0.2t}$, but different in one important way. Where we took the natural logarithm of both sides in that example, we would need to take $\log_{1.0125}$ of both sides here! Instead, we take the common logarithm $(\log_{10})$ of both sides and apply the rule

$$\log_a u^r = r\,\log_a u$$

that we saw in the last section. Here's how it's done:

◇ **Example 6.6(a):** Solve the equation $5000 = 1000(1.0125)^{4t}$.

5000	$=$	$1000(1.0125)^{4t}$	the original equation
5	$=$	1.0125^{4t}	divide both sides by 1000
$\log 5$	$=$	$\log 1.0125^{4t}$	take the common logarithm of both sides
$\log 5$	$=$	$4t\log 1.0125$	apply the rule $\log_a u^r = r\cdot\log_a u$
$\dfrac{\log 5}{4\log 1.0125}$	$=$	t	divide both sides by $4\log 1.0125$
32.39	$=$	t	compute the final solution

It will take about 32.4 years for an investment of $\$1000$ to reach a value of $\$5000$ when invested at 5%, compounded quarterly.

Note that we could have used the natural logarithm everywhere that the common logarithm was used above, but we'll generally save the natural logarithm for situations dealing with the number e.

Remember the change of base formula $\log_a B = \dfrac{\log B}{\log a}$? Well, now we are ready to see where this comes from. Suppose that we want to find $\log_a B$; we begin by setting $x = \log_a B$. If we then solve this equation for x in the same way as just done, we'll have the value of $\log_a B$:

$$\begin{aligned}
x &= \log_a B && \text{the original equation}\\
a^x &= B && \text{apply the definition of a logarithm}\\
& && \text{to change to exponential form}\\
\log a^x &= \log B && \text{take the common logarithm of both sides}\\
x \log a &= \log B && \text{apply the rule } \log_a u^r = r \cdot \log_a u\\
x &= \frac{\log B}{\log a} && \text{divide both sides by } \log a
\end{aligned}$$

Creating Exponential Models

Recall, from Section 6.2, the general forms of exponential growth and exponential decay models:

> **Exponential Growth and Decay Model**
>
> When a quantity starts with an amount A_0 or a number N_0 and grows or decays exponentially, the amount A or number N after time t is given by
>
> $$A = A_0 e^{kt} \qquad \text{or} \qquad N = N_0 e^{kt}$$
>
> for some constant k, called the **growth rate** or **decay rate**. When k is a growth rate it is positive, and when it is a decay rate it is negative.

The variables in the above models are N and t, or A and t, depending on which model is being used. (You should be able to see by now that they are really the same model!) The values N_0, A_0 and k are what we call parameters. They vary from situation to situation, but for a given situation they remain constant. When working with any exponential growth or decay problem the first thing we need to do is determine the values of these parameters, if they are not given to us. The next example shows how this is done.

◇ **Example 6.6(b):** A new radioactive element, *Watermanium*, is discovered. You have a sample of it that you know weighed 54 grams when it was formed, and there are 23 grams of it 7 days later. Determine the equation that models the decay of *Watermanium*.

Solution: We use the model $A = A_0 e^{kt}$, knowing that $A_0 = 54$. Because there are 23 grams left at 7 days, we can put $A = 23$ and $t = 7$ into the equation and it must be true: $23 = 54e^{7k}$. Solving this equation for k gives us $k = -0.12$. Our model is then $A = 54e^{-0.12t}$.

Note that the constant k depends on the material, in this case *Watermanium*, and A_0 is the initial amount, which can of course change with the scenario.

Section 6.6 Exercises To Solutions

1. Solve $1500 = 500(2)^{5t}$ for t. Round to the hundredth's place.

2. In Example 6.6(b) we found that the amount of *Watermanium* left in a 54 gram sample after t days is given by $A = 54e^{-0.12t}$.

   (a) To the nearest gram, how much *Watermanium* will be left in 10 days?

   (b) How many days would it take a 347 gram sample of *Watermanium* to decay to 300 grams? Round your answer to the nearest tenth of a day. (**Hint:** The decay constant $k = -0.12$ does not depend on the original amount or its units.)

   (c) What is the half-life of Watermanium? Round to the nearest tenth of a day.

   (d) How many days, to the nearest tenth, will it take any amount of *Watermanium* to decay to 10% of the original amount?

3. Recall the culture of bacteria from Example 6.2(a), whose growth model was the function $f(t) = 60(3)^{t/2}$. Find the length of time that it would take for the number of bacteria to grow to 4000. Give your answer to the nearest hundredth of an hour.

4. Suppose that you invest some money at an annual percentage rate of 8%, compounded semi-annually. How long would it take for you to double your money?

5. The population of a certain kind of fish in a lake has a doubling time of 3.5 years. A scientist doing a study estimates that there are currently about 900 fish in the lake.

   (a) Using only the the doubling time, estimate when there will be 5000 fish in the lake. Do this by saying that "It will take between _______ and _______ years for there to be 5000 fish in the lake."

   (b) Create an exponential model for the fish population. Round k to the thousandth's place.

   (c) Use your model from (b) to determine when, to the nearest tenth of a year, the fish population in the lake will reach 5000. Check with your answer to (a).

6. Use the half-life of 5.8 days from Exercise 2(c) to find a function of the form $W = C(2)^{kt}$ that gives the amount of *Watermanium* left after t days in a sample that was originally 37.5 ounces.

7. Solve $1500 = 500(2)^{5t}$ for t, **to the nearest hundredth**.

8. Suppose that you invest $1500 at an annual percentage rate of 8%, compounded semi-annually. How long would it take for you to double your money?

9. If we begin with 80 milligrams (mg) of radioactive ^{210}Bi (bismuth), the amount A after t days is given by $A = 80(2)^{-t/5}$. Determine when there will be 12 milligrams left.

10. You invest some money at an annual interest rate of 5%, compounded quarterly. How long will it take to triple your money?

11. A culture of bacteria began with 1000 bacteria. An hour later there were 1150 bacteria. Assuming exponential growth, how many will there be 12 hours after starting? (**Hint:** You will need to create an exponential model.) Round to the nearest whole bacterium.

12. A 25 pound sample of radioactive material decays to 24.1 pounds in 30 days. How much of the original will be left after two years?

13. A drug is eliminated from the body through urine. Suppose that for an initial dose of 20 milligrams, the amount $A(t)$ in the body t hours later is given by $A(t) = 20(0.85)^t$. Determine when there will be 1 milligram left.

7 Systems of Equations

Outcome/Performance Criteria:

7. Solve systems of equations. Apply systems of equations to solve problems.

   (a) Use the addition method to solve a system of three linear equations in three unknowns.

   (b) Row-reduce a matrix.

   (c) Solve a system of equations by row-reduction, without using a calculator, and using a calculator.

   (d) Know when it is possible to add, subtract or multiply two matrices. Add, subtract and multiply matrices. Multiply matrices by numbers.

   (e) Determine whether two matrices are inverses without computing the inverse of either; use the formula to find the inverse of a 2×2 matrix.

   (f) Write a system of equations as a matrix equation.

   (g) Solve a system of equations using the inverse matrix method, with or without a calculator.

   (h) Calculate the determinant of a 2×2 matrix; use Cramer's rule to solve a system of two equations in two unknowns.

7.1 Systems of Three Linear Equations

Performance Criteria:

7. (a) Use the addition method to solve a system of three linear equations in three unknowns.

Recall the following Example from Chapter 1:

◇ **Example 1.6(a):** Solve the system $\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$ using the addition method.

Solution: The idea is to be able to add the two equations and get either the x terms or the y terms to go away. If we multiply the first equation by 5 and the second equation by 4 the y terms will go away when we add them together. We'll then solve for x:

$$\begin{aligned} 2x - 4y = 18 &\overset{\text{times } 5}{\Longrightarrow} & 10x - 20y = 90 \\ 3x + 5y = \ \ 5 &\overset{\text{times } -4}{\Longrightarrow} & \underline{12x + 20y = 20} \\ & & 22x = 110 \\ & & x = 5 \end{aligned}$$

We then substitute this value of x into any one of our equations to get y:

$$\begin{aligned} 3(5) + 5y &= \ \ 5 \\ 15 + 5y &= \ \ 5 \\ 5y &= -10 \\ y &= -2 \end{aligned}$$

The solution to the system is then $x = 5,\ y = -2,$ or $(5, -2)$.

We should note that the system could have been solved by eliminating x, rather than y:

◇ **Example 7.1(a):** Solve the system $\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$ using the addition method.

Solution: If we multiply the first equation by 3 and the second equation by -2 the x terms will go away when we add them together. We'll then solve for y:

$$\begin{aligned} 2x - 4y = 18 &\overset{\text{times } 3}{\Longrightarrow} & 6x - 12y = \ \ 54 \\ 3x + 5y = \ \ 5 &\overset{\text{times } -2}{\Longrightarrow} & \underline{-6x - 10y = -10} \\ & & -22y = 44 \\ & & y = -2 \end{aligned}$$

We then substitute this value of y into any one of our equations to get $x = 5$.

In Section 1.6 you may have solved the following system of *three equations in three unknowns*:

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \qquad (1)$$

The sequence we use for such a system is the following:

- Add a multiple of the first equation to a multiple of the second equation in order to eliminate x.
- Add a multiple of the first equation to a multiple of the third equation equation in order to eliminate x.
- The first two steps results in a system of two equations in the unknowns y and z. Add a multiple of one to a multiple of the other in order to eliminate y, and solve for z.
- Substitute the value of z into either of the equations containing only y and z and solve for y.
- Substitute the values of y and z into any of the three original equations and solve for x.

Let's demonstrate:

◇ **Example 7.1(b):** Solve the above system of three equations in three unknowns.

Solution: Multiplying the first equation by -3 and adding it to the second equation results in the equation $-2y+7z=16$. Similarly, multiplying the first equation by 2 and adding to the third equation gives $7y+3z=-1$. We now have the system

$$\begin{aligned} -2y + 7z &= 16 \\ 7y + 3z &= -1 \end{aligned}$$

of two equations in the unknowns y and z. Multiplying the first of these by 7, the second by 2 and adding results in $55z = 110$. From this we determine that $z = 2$ and, substituting into either of the above two equations and solving for y gives us $y = -1$. Finally, we substitute our known values for y and z into any of the first three equations and solve to get $x = 3$.

Section 7.1 Exercises To Solutions

1. Consider the system of three equations in three unknowns x, y and z:

$$\begin{aligned} x+2y-z &= -1 \\ 2x-y+3z &= 13 \\ 3x-2y &= 6 \end{aligned}$$

Follow the steps below to solve the system.

(a) Use the addition method with the first two equations to eliminate x.

(b) Use the addition method with the first and third equations to eliminate x.

(c) Your answers to (a) and (b) are a new system of two equations with two unknowns. Use the addition method to eliminate y and solve for z.

(d) Substitute the value you found for z into one of the equations containing y and z to find y.

(e) You should now know y and z. Substitute them into ANY of the three equations to find x.

(f) You now have what we call an *ordered triple* (x, y, z). Check your solution by substituting those three numbers into each of the original equations to make sure that they make all three equations true.

2. Solve each of the following systems of equations.

(a) $$\begin{aligned} x+3y-z &= -3 \\ 3x-y+2z &= 1 \\ 2x-y+z &= -1 \end{aligned}$$

(b) $$\begin{aligned} 2x-y+z &= 6 \\ 4x+3y-z &= 1 \\ -4x-8y+2z &= 1 \end{aligned}$$

(c) $$\begin{aligned} x-2y+3z &= 4 \\ 2x+y-4z &= 3 \\ -3x+4y-z &= -2 \end{aligned}$$

7.2 Solving Systems of Equations by Row-Reduction

Performance Criteria:

7. (b) Row-reduce a matrix.
   (c) Solve a system of equations by row-reduction, without using a calculator, and using a calculator.

In Example 7.1(b) we solved the system of equations

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \qquad (1)$$

If we look carefully at our work there we find the following sequence of systems of equations:

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x+3y-2z &= -4 \\ -2y+7z &= 16 \\ 7y+3z &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x+3y-2z &= -4 \\ -2y+7z &= 16 \\ 55z &= 110 \end{aligned} \qquad (2)$$

Let's put in zeros for coefficients of missing terms:

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x+3y-2z &= -4 \\ 0x-2y+7z &= 16 \\ 0x+7y+3z &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x+3y-2z &= -4 \\ 0x-2y+7z &= 16 \\ 0x+0y+55z &= 110 \end{aligned} \qquad (3)$$

It turns out that the symbols x, y, z and $=$ just get in the way while we are solving the system of equations. Because of this it is convenient to arrange just the coefficients of the unknowns and the numbers from (1) on the right sides into a table of values called a **matrix**

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \qquad (4)$$

This is called the **augmented matrix** for the system (1). The goal is to manipulate this matrix to obtain the matrix

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 55 & 110 \end{bmatrix} \qquad (5)$$

Note that this is just the matrix for the last system in (3), which is really the same as the last system in (2).

To get from (4) to (5) we use a process called **row reduction**. When working with a matrix that represents a system of equations we are allowed to do three things without affecting the solution to the system:

- Multiply each entry in a row by the same value. (We can also divide by a number, which is the same as multiplying by its reciprocal.)
- Add a row to another row (replacing the second of these two rows with the result in the process). (We can also subtract a row from another, since that is the same as first multiplying by -1 and then adding.)
- Rearrange the rows.

What we will do most often is a combination: We'll multiply a row by a number and add it to another row. To begin getting from (4) to (5) we first multiply the first row of (4) by -3 and add the result to the second row, replacing the second row with the result. We will symbolize this by

$$-3R_1 + R_2 \to R_2.$$

This says "add negative three times row one to row two, and put the result in row two." *Leave the first row in its original state.* This is how we show it and what we get:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \quad \underset{-3R_1 + R_2 \to R_2}{\Longrightarrow} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ -2 & 1 & 7 & 7 \end{bmatrix} \qquad (6)$$

Next we multiply the first row by 2 and add the result to the third row, putting the result in the third row, leaving the first two rows as they were. Often we will combine these first two steps into one:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \quad \begin{array}{c} -3R_1 + R_2 \to R_2 \\ \Longrightarrow \\ 2R_1 + R_3 \to R_3 \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix} \qquad (7)$$

Now we multiply the second row by 7 and the third row by 2 and add the results, putting what we get in the third row, leaving the other two rows the same. The result should be (5) above. The symbolic representation of this is

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix} \quad \underset{7R_2 + 2R_3 \to R_3}{\Longrightarrow} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 55 & 110 \end{bmatrix}$$

At this point we have **row-reduced** the matrix (4) to the *row-reduced form* (5). It is possible to take this process farther to give a nicer result to work with, but it is generally not worth the effort to do so. We can get the solution to the system by first putting the final matrix (5) back into equation form:

$$\begin{aligned} x + 3y - 2z &= -4 \\ 0x - 2y + 7z &= 16 \\ 0x + 0y + 55z &= 110 \end{aligned}$$

We then solve the last equation for $z = 2$. Putting that value into the second equation and solving, we get $y = -1$, Finally we substitute our y and z values into the first equation to get $x = 3$. The full solution is therefore $(3, -1, 2)$. In Section 7.1 we saw that the solution to a system of two equations in two unknowns is a point in the plane where the two lines represented by the equations cross each other. When we deal with three linear equations in three unknowns, each equation represents a plane in three-dimensional space, and the three numbers in the solutions are the coordinates of the point in three-dimensional space where the three planes intersect.

In a moment we will see how to let our calculator do all the previous calculations for us. The calculator will take our final matrix above and continue, to obtain ones where we have 1, -2 and 55 and zeros in all the places above them. The process goes like this:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 55 & 110 \end{bmatrix} \underset{\frac{1}{55}R_3 \to R_3}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{c} -7R_3 + R_2 \to R_2 \\ \Longrightarrow \\ 2R_3 + R_1 \to R_1 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{-\frac{1}{2}R_2 \to R_2}{\Longrightarrow} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{-3R_2 + R_1 \to R_1}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This last matrix is equivalent to the system

$$\begin{aligned} x + 0y + 0z &= 3 \\ 0x + y + 0z &= -1 \\ 0x + 0y + z &= 2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= 3 \\ y &= -1 \\ z &= 2 \end{aligned}$$

Row Reduction Using Your Calculator

The process of row reduction is quite tedious, and the systems of equations that I have given you so far are easier to work with than most. Let's see how to make our calculators do the "dirty work!" Your calculator is going to take the row-reduction process as far as we just did, so you will be able to just read off the values of x, y and z. As you read this you should go through the process with the system we've been working with,

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \tag{1}$$

- Select *MATRIX* (or maybe *MATRX*) somewhere.
- Select *EDIT*. At that point you will see something like 3×3 somewhere. This is the number of rows and columns your matrix is going to have. We want 3×4.
- After you have told the calculator that you want a 3×4 matrix, it will begin prompting you for the entries in the matrix, starting in the upper left corner. Here you will begin entering the values from (4), row by row. You should see the entries appear in a matrix as you enter them.

- After you enter the matrix, you need to select *MATH* under the *MATRIX* menu. Select *rref* (this stands for **row-reduced echelon form**) and you should see *rref* (on your calculator screen.
- Select *NAMES* under the *MATRIX* menu. Highlight A and hit enter, then enter again. At that point you should see the matrix(8), from which the solution can be read off.

Section 7.2 Exercises To Solutions

1. Fill in the blanks in the second matrix with the appropriate values after the first step of row-reduction. Fill in the long blanks with the row operations used.

(a)

$$\begin{bmatrix} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{bmatrix} \quad \begin{array}{c} \underline{\hspace{4cm}} \\ \Longrightarrow \\ \underline{\hspace{4cm}} \end{array} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 3 & 1 \\ 0 & 6 & -5 & 2 \end{bmatrix} \quad \begin{array}{c} \Longrightarrow \\ \\ \underline{\hspace{4cm}} \end{array} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & 0 & \_\_ & \_\_ \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{bmatrix} \quad \begin{array}{c} \Longrightarrow \\ \\ \underline{\hspace{4cm}} \end{array} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & 0 & \_\_ & \_\_ \end{bmatrix}$$

2. Find x, y and z for the system of equations that reduces to the each of the matrices shown.

(a) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & -2 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 2 & -5 & -13 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 8 \end{bmatrix}$

3. Use row operations on an augmented matrix to solve each system of equations.

(a) $\begin{aligned} x - 2y - 3z &= -1 \\ 2x + y + z &= 6 \\ x + 3y - 2z &= 13 \end{aligned}$ (b) $\begin{aligned} -x - y + 2z &= 5 \\ 2x + 3y - z &= -3 \\ 5x - 2y + z &= -10 \end{aligned}$

(c) $\begin{aligned} x + 2y + 4z &= 7 \\ 2x + 3y + 3z &= 7 \\ -x + y + 2z &= 5 \end{aligned}$

4. Use your calculator to solve each of the systems of equations from Exercise 3.

7.3 The Algebra of Matrices

Performance Criteria:

7. (d) Know when it is possible to add, subtract or multiply two matrices. Add, subtract and multiply matrices. Multiply matrices by numbers.

In this section we see how to add, subtract and multiply matrices. *It is not possible to divide one matrix by another.* Before beginning we need to know a tiny bit of language of matrices. The individual numbers in a matrix are called the **entries** of the matrix. The **rows** and **columns** of a matrix should be obvious - the rows are the *horizontal* rows of entries, and the columns are the *vertical* columns of entries. The numbers of rows and columns of a matrix, *in that order*, are the **dimensions** of the matrix. For example, a matrix with 3 rows and four columns has dimensions 3×4, which we read as "three by four."

I will use the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

to demonstrate the algebra of matrices. (*Note that we use CAPITAL letters to name matrices.*) Matrices like these, with the same number of rows as columns, are called **square matrices**. *Note that we use CAPITAL letters to name matrices.* I am using letters rather than numbers for the entries of the matrices because it will make it easier to follow what is happening. First we show how to add and subtract two matrices:

$$A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \qquad A - B = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

In other words, we add and subtract matrices "entry by entry". We should make note of a couple things:

- To add or subtract two matrices, they must have the same dimensions.
- Whenever we are using an operation to put two mathematical objects together to form one, we should always ask whether the order of the two objects matters. If it doesn't, we say the operation is **commutative**. As with numbers, addition of matrices is commutative and subtraction is not.

We can also multiply a matrix by a number - we simply multiply every entry of the matrix by that number. For example,

$$5A = 5\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5a & 5b \\ 5c & 5d \end{bmatrix}$$

◇ **Example 7.3(a):** For $A = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix}$, find $-A + 2B$.

$$-A + 2B = -\begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} + 2\begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -1 & -8 \end{bmatrix} + \begin{bmatrix} 6 & 14 \\ -8 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ -9 & -8 \end{bmatrix}$$

Adding and subtracting matrices is neither interesting nor very useful in applications. Much more important (and more complicated to do) is multiplying two matrices. I will try to describe how to do it here, but it is best learned "in person". Consider again the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad\qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

We will find that *matrices do not have to be the same dimensions to multiply them. There ARE certain other restrictions on their dimensions in order to be able to multiply them, however.* The only time we can multiply two matrices with the same dimensions is when they are both square as well, and the resulting matrix is also square with those dimensions. When we multiply A and B we will get a 2 by 2 matrix as a result.

Since our result is going to be a 2 by 2 matrix, it will have four entries. The question is then "How do we get each entry of the resulting matrix?" To get the first entry we "multiply" the first *row* of the first matrix by the first *column* of the second matrix. Actually we multiply the entries (two multiplications) and add the results:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & * \\ * & * \end{bmatrix}.$$

Here $*$ simply means an entry that is not being shown, to avoid cluttering things up!

To get the second entry (second entry of the first row) we multiply the first row of the first matrix (again) by the *second* column of the second matrix:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ * & * \end{bmatrix}.$$

To get the second row of the resulting matrix we multiply the second row of the first matrix by the first and second columns of the second matrix, respectively. The final result is

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

Let's look at a specific example.

◇ **Example 7.3(b):** Consider $A = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix}$. Find AB and BA.

$$AB = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} (-5)(3) + (2)(-4) & (-5)(7) + (2)(0) \\ (1)(3) + (8)(-4) & (1)(7) + (8)(0) \end{bmatrix} = \begin{bmatrix} -23 & -35 \\ -29 & 7 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} (3)(-5)+(7)(1) & (3)(2)+(7)(8) \\ (-4)(-5)+(0)(1) & (-4)(2)+(0)(8) \end{bmatrix} = \begin{bmatrix} -8 & 62 \\ 20 & -8 \end{bmatrix}$$

There are two very important comments to be made at this point:

- As can bee seen in this example, **matrix multiplication is not generally commutative!** That is, in most cases $AB \neq BA$. In fact it is sometimes possible to compute only one of AB or BA, but not the other.
- Although it is not true in general that $AB = BA$, it *can* be true sometimes. You will see some examples soon.

A matrix with the same number of rows as columns is called a **square matrix**. All of our examples so far have been square matrices. If we have a square matrix A, it can be multiplied by itself, so for a square matrix A it makes sense to talk about A^2. *This is only meaningful for square matrices!*

◇ **Example 7.3(c):** If $A = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}$, find A^2.

$$A^2 = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} (-5)(-5)+(2)(1) & (-5)(2)+(2)(8) \\ (1)(-5)+(8)(1) & (1)(2)+(8)(8) \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ 3 & 66 \end{bmatrix}$$

The next example looks at when matrices can be multiplied and when they cannot.

◇ **Example 7.3(d):** For the following matrices, give an example of each of the following:

- Two non-square matrices that can be multiplied.
- A square matrix and a non-square matrix that can be multiplied.
- Two matrices that cannot be multiplied.
- A matrix that can be squared.
- A matrix that cannot be squared.

$$A = \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} -5 \\ 4 \\ -7 \end{bmatrix}$$

Solution: B and C are both non-square, and BC exists. Note also that CB does not exist. This takes care of the first and third examples. A is square, B is non-square, and AB exists. Finally, A can be squared, and B and C cannot.

◇ **Example 7.3(e):** For the matrices A and B from the previous example, find AB.

$$\begin{aligned} AB &= \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (0)(4)+(5)(-2) & (0)(-7)+(5)(0) & (0)(3)+(5)(5) \\ (-3)(4)+(1)(-2) & (-3)(-7)+(1)(0) & (-3)(3)+(1)(5) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

$$IB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (1)(a)+(0)(c) & (1)(b)+(0)(d) \\ (0)(a)+(1)(c) & (0)(b)+(1)(d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If you understand how to multiply two matrices, you should be able to determine whether two matrices can be multiplied without relying on any rules. You should also be able to determine the dimensions of the resulting matrix. However, to summarize the facts:

> If matrix A has dimensions $m \times n$ and matrix B has dimensions $p \times q$, then AB exists only if $n = p$. That is, *the number of columns in the first matrix must equal the number of rows in the second.* In that case, the matrix AB has dimensions $m \times q$.

◇ **Example 7.3(f):** Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find BI and IB.

$$BI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (a)(1)+(b)(0) & (a)(0)+(b)(1) \\ (c)(1)+(d)(0) & (c)(0)+(d)(1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$IB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (1)(a)+(0)(c) & (1)(b)+(0)(d) \\ (0)(a)+(1)(c) & (0)(b)+(1)(d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The matrix I from this last example is called the 2×2 **identity matrix**. In math, an identity is something that can be combined, via some operation, with all other things without changing them. Technically speaking, we must give the operation as well as the identity:

- For numbers, zero is the *additive* identity because $a + 0 = 0 + a = a$ for every number a.
- For numbers, one is the *multiplicative* identity because $a \cdot 1 = 1 \cdot a = a$ for every number a.
- For matrices, the matrix O of all zeros (called the zero matrix) is the additive identity matrix because it can be added to any other matrix (of the same dimensions) without changing it.

It turns out that the additive identity for matrices has little or no use, so when we are working with matrices and talk about the identity matrix, we mean multiplicative identity. There are many identity matrices of various sizes, but each satisfies the following:

- They are square matrices. (There is then *one* identity matrix of each size of square matrix.)
- Their entries consist of ones on the diagonal (from upper left to lower right) and zeroes elsewhere.
- For any square matrix A of the same size as I, $AI = IA = A$.

Even though the last statement is about square matrices A, the products AI and IA always equal A *whenever they can be carried out*. However, if A is not square each of these products will use a different sized identity matrix. It will not be immediately apparent to you why we care about the identity matrices, but we do!

Section 7.3 Exercises To Solutions

1. Consider the matrices shown below. There are *THIRTEEN* multiplications possible, including squaring. Find and compute each possible multiplication *by hand*. (**Hint:** Be systematic - try A times each matrix (including itself, since you are looking for matrices that can be squared also), *in that order*. Then try B times every other matrix (including A, since AB is not necessarily equal to BA), in order, and so on.)

$$A = \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} -5 \\ 4 \\ -7 \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 3 \\ -5 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 5 & -1 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & -1 \\ 6 & 9 \end{bmatrix}$$

2. Fill in the blanks:

$$\begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & -7 \\ -5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 8 & -2 \\ -1 & -3 & 5 \\ 6 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \underline{\qquad} & * & * \\ * & * & \underline{\qquad} \\ * & * & * \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find AI and IA.

4. Let $A = \begin{bmatrix} 3 & 0 & 2 \\ -1 & 4 & 5 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find AB and AX.

5. Let $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$. Find$AC$ and CA.

7.4 Solving Systems of Equations With Inverse Matrices

Performance Criteria:

7. (e) Determine whether two matrices are inverses without computing the inverse of either; use the formula to find the inverse of a 2×2 matrix.

   (f) Write a system of equations as a matrix equation.

   (g) Solve a system of equations using the inverse matrix method, with or without a calculator.

Inverse Matrices

Let's begin with an example.

◇ **Example 7.4(a):** Find AC and CA for $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$.

$$AC = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} (5)(3)+(7)(-2) & (5)(-7)+(7)(5) \\ (2)(3)+(3)(-2) & (2)(-7)+(3)(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$CA = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} (3)(5)+(-7)(2) & (3)(7)+(-7)(3) \\ (-2)(5)+(5)(2) & (-2)(7)+(5)(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

What just occurred is special; it is very much like the fact that $\frac{1}{3}(3) = 1$, the multiplicative identity. The fact that $\frac{1}{3}(3) = 1$ is because $\frac{1}{3}$ and 3 are **multiplicative inverses**. In the above example the matrices A and C are multiplicative inverses as well, since their product is the multiplicative identity.

Definition of Inverse Matrices

Suppose that for matrices A and B we have $AB = BA = I$. Then we say that A and B are **inverse matrices**.

Notationally we write $B = A^{-1}$ or $A = B^{-1}$. Note that in order for us to be able to do both multiplications AB and BA, both matrices must be square and of the same dimensions. It also turns out that that to test two square matrices to see if they are inverses we only need to multiply them in one order:

Test for Inverse Matrices

To test two *square* matrices A and B to see if they are inverses, compute AB. If it is the identity, then the matrices are inverses.

Here are a few notes about inverse matrices:

- Not every square matrix has an inverse, but many do. If a matrix does have an inverse, it is said to be **invertible**.
- The inverse of a matrix is unique, meaning there is only one.
- Matrix multiplication *IS* commutative for inverse matrices.

Two questions that should be occurring to you now are

1) How do we know whether a particular matrix has an inverse?
2) If a matrix does have an inverse, how do we find it?

There are a number of ways to answer the first question; here is one:

> **Test for Invertibility of a Matrix**
>
> A square matrix A is invertible if *rref* $(A) = I$.

Here is the answer to the second question in the case of a 2×2 matrix:

> **Inverse of a 2×2 Matrix**
>
> The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For square matrices larger than 2×2 there is a fairly simple method for finding the inverse that we will not go into here.

◇ **Example 7.4(b):** Find the inverse of $A = \begin{bmatrix} -2 & 7 \\ 1 & -5 \end{bmatrix}$.

$$A^{-1} = \frac{1}{(-2)(-5) - (1)(7)}\begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Matrices and Systems of Equations

Once again we begin with an example.

◇ **Example 7.4(c):** Let $A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find AX

$$AX = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 3y + z \\ 2x + 5y \\ 3x + y - 2z \end{bmatrix}$$

Because of what you have just seen, a system of equations like

$$\begin{aligned} -1x + 3y \ + \ z &= 2 \\ 2x + 5y \qquad &= 1 \\ 3x + \ y - 2z &= -4 \end{aligned}$$

can be written as

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

or $AX = B$, where

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

The following summarizes this for a system of three linear equations in three unknowns, but of course the same thing can be done for larger systems as well.

Matrix Equation Form of a System

A system of three linear equations in three unknowns can be written as $AX = B$ where A is the 3×3 **coefficient matrix** of the system, X is the 3×1 matrix consisting of the three unknowns and B is the 3×1 matrix consisting of the right-hand sides of the equations, as shown below.

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned} \iff \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

A system of two equations in two unknowns can be written the same way, except that A will then be 2×2 and X and B will be 2×1.

This form of a system of equations can be used, as you will soon see, in another method (besides row-reduction) for solving a system of equations.

⋄ **Example 7.4(d):** Give the matrix form of the system
$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+\ z &= 4 \\ -2x+\ y+7z &= 7 \end{aligned}$$

Solution: The matrix form of the system is $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$

Suppose now that we have a system of equations and put it in the matrix form $AX = B$. We then solve it in much the same way that we would solve $3x = 5$, but with one small difference. Where we would ordinarily divide both sides of $3x = 5$ by 3, we can accomplish the same thing by multiplying by $\frac{1}{3}$, *the multiplicative inverse of* 3. In the matrix version we multiply by A^{-1}, the multiplicative inverse of A. Let's do the two side-by-side:

$$\begin{aligned} 3x &= 5 \\ \tfrac{1}{3}\cdot 3x &= \tfrac{1}{3}\cdot 5 \\ 1x &= \tfrac{5}{3} \\ x &= \tfrac{5}{3} \end{aligned} \qquad\qquad \begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

⋄ **Example 7.4(e):** Use the above method to solve the system
$$\begin{aligned} 5x+4y &= 25 \\ -2x-2y &= -12 \end{aligned}.$$

Solution: The matrix form of the system is $\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 25 \\ -12 \end{bmatrix}$, so $A = \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}$.

We can use the formula for the inverse of a 2×2 matrix to find

$$A^{-1} = \frac{1}{(5)(-2)-(-2)(4)}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -\frac{5}{2} \end{bmatrix}$$

We can now solve the matrix equation by multiplying both sides by A^{-1} *on the left.*

$$\begin{aligned} \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 25 \\ -12 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ -1 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ -1 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 25 \\ -12 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{aligned}$$

The solution to the system is $x = 1$, $y = 5$ or $(1,5)$.

Section 7.4 Exercises To Solutions

1. (a) Determine whether $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$ are inverses without finding the inverse of either.

   (b) Determine whether $A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix}$ and $B = \frac{1}{14}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 14 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ are inverses without finding the inverse of either.

2. As you just saw, a system of equations can be written in the form $AX = B$ using matrices. For each of the following systems of equations, determine the matrices A, X and B.

   (a) $\begin{aligned} 3x + 2y &= -1 \\ 4x + 5y &= 1 \end{aligned}$ (b) $\begin{aligned} -2x + 2y + 3z &= -1 \\ x - y \quad &= 0 \\ y + 4z &= 4 \end{aligned}$ (c) $\begin{aligned} x + 2y + 3z &= -3 \\ -2x + y \quad &= -2 \\ 3x - y + z &= 1 \end{aligned}$

3. Solve each of the following systems using the method of Example 7.6(e), showing all steps in the manner that was done there. Use the formula for the inverse of a 2×2 matrix to find A^{-1} in each case, but from step two to step three be sure to multiply $A^{-1}A$ to make sure that you calculated the inverse correctly.

   (a) $\begin{aligned} 5x + 7y &= -4 \\ 2x + 3y &= 1 \end{aligned}$ (b) $\begin{aligned} 3x + 2y &= -1 \\ 4x + 5y &= 1 \end{aligned}$ (c) $\begin{aligned} 2x + 5y &= 3 \\ 3x + 8y &= -2 \end{aligned}$

4. Use the information below to solve each of the following systems of equations using inverse matrices. Do these by hand; you can go straight to the solution $X = A^{-1}B$, but do the computation first. (**Hint:** Multiply the matrices first, then multiply by the number.)

   (a) $\begin{aligned} -1x + 3y + z &= 2 \\ 2x + 5y \quad &= 1 \\ 3x + y - 2z &= -4 \end{aligned}$ (b) $\begin{aligned} -2x + 2y + 3z &= -1 \\ x - y \quad &= 0 \\ y + 4z &= 4 \end{aligned}$ (c) $\begin{aligned} x + 2y + 3z &= -3 \\ -2x + y \quad &= -2 \\ 3x - y + z &= 1 \end{aligned}$

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -5 & -3 \\ 2 & -8 & -6 \\ -1 & 7 & 5 \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{9}\begin{bmatrix} -10 & 7 & -5 \\ 4 & -1 & 2 \\ -13 & 10 & -11 \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

7.5 Determinants and Cramer's Rule

Performance Criteria:

7. (h) Calculate the determinant of a 2×2 matrix; use Cramer's rule to solve a system of two equations in two unknowns.

Let's consider the system of equations $\begin{array}{rcl} 5x+3y & = & 1 \\ 4x+2y & = & 2 \end{array}$. To solve for x we can multiply the first equation by 2 and the second equation by -3 to obtain

$$\begin{array}{rcl} 10x+6y & = & 2 \\ -12x-6y & = & -6 \end{array}.$$

We then add the two equations to obtain the equation $-2x=-4$, so $x=2$. Note that we could instead multiply the first equation by 2, the second by 3, and then *subtract* the two equations to get

$$\begin{array}{rcr} 5x+3y=1 & \Longrightarrow & 10x+6y=2 \\ 4x+2y=2 & \Longrightarrow & \underline{12x+6y=6} \\ & & -2x=-4, \end{array}$$

solving the resulting equation to get $x=2$. Using this process on a general system of two equations we get

$$\begin{array}{rcl} ax+by=e & \Rightarrow & adx+bdy=ed \\ cx+dy=f & \Rightarrow & \underline{bcx+bdy=bf} \\ & & adx-bcx=ed-bf \end{array} \quad\Rightarrow\quad (ad-bc)x=ed-bf \quad\Rightarrow\quad x=\frac{ed-bf}{ad-bc} \qquad (1)$$

We can also multiply the top equation by c and the bottom equation by a and then subtract the top equation from the bottom one to get

$$\begin{array}{rcl} ax+by=e & \Rightarrow & acx+bcy=ce \\ cx+dy=f & \Rightarrow & \underline{acx+ady=af} \\ & & ady-bcy=af-ce \end{array} \quad\Rightarrow\quad (ad-bc)y=af-ce \quad\Rightarrow\quad y=\frac{af-ce}{ad-bc} \qquad (2)$$

Note the matrix form of the system of equations:

$$\begin{array}{r} ax+by=E \\ cx+dy=f \end{array} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} e \\ f \end{bmatrix}.$$

If we look carefully at the expressions for x and y that were obtained in (1) and (2) above, we see that they both have the same denominator $ad-bc$, and that the numbers a, b, c and d are the entries in the coefficient matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We will return to (1) and (2) after introducing a new concept called the determinant of a matrix.

Determinants of Matrices

There are situations in math where we attach a number to an object, with that number somehow giving us some information about the object. An example you are familiar with is that every non-vertical line has a number attached to it, its slope. That number describes the steepness of the line (and whether it slopes upward or downward as we go left to right). There is a method for finding the slope of any line, which essentially boils down to "rise over run."

Every *square* matrix has a number associated with it called its **determinant**. The method for computing the determinant of a 2×2 matrix is fairly straightforward:

> **Determinant of a 2×2 Matrix**
>
> The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

Suppose that we have the square matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$. It should be clear from the definition that the determinant of this matrix is $(3)(2) - (2)(-1) = 17$. There are two ways we generally indicate the determinant of a matrix: the first is to simply write $\det$ before the matrix, and the second is to replace the brackets $[\ ]$ around the matrix with vertical bars $|\ |$. So for our matrix we could write

$$\det \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} = 17 \qquad \text{or} \qquad \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = 17.$$

Thus $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ is the matrix itself, whereas $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$ is the determinant of the matrix, a single number.

Finding determinants of 3×3 matrices by hand is a little complicated, and finding determinants of larger matrices is yet more complicated. They are easily found by graphing calculators or online tools, however.

Now what does the determinant tell us a bout matrix? Well, here is the most basic fact about determinants:

> **Determinants and Invertibility**
>
> A square matrix has an inverse if, and only if, its determinant is *NOT* zero.

One way of interpreting the above is that *if the determinant of a matrix is zero, then the matrix does not have an inverse*. Now recall the following computations from the previous section:

$$\begin{aligned} 3x &= 5 \\ \tfrac{1}{3} \cdot 3x &= \tfrac{1}{3} \cdot 5 \\ 1x &= \tfrac{5}{3} \\ x &= \tfrac{5}{3} \end{aligned} \qquad\qquad \begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Note that is our original equation on the left had been $0x = 5$ we would not have been able to use the method shown because $\frac{1}{0}$ is undefined. Similarly, if A does not have an inverse, we can't solve the equation $AX = B$ in the way shown above and to the right. Therefore, the matrices with determinant zero (and there are infinitely many of them) are to all other matrices as the number zero is to all other numbers.

Cramer's Rule

We will now see what determinants have to do with results (1) and (2) from earlier in this section. Recall that we had the system of equations with the standard and matrix forms

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

The solution to the system was given by the two expressions

$$x = \frac{ed - bf}{ad - bc} \qquad \text{and} \qquad y = \frac{af - ce}{ad - bc}.$$

We can now see that the denominator of each of the expressions is the determinant of the coefficient matrix of the system, but what are the numerators? They also appear to be determinants of 2×2 matrices. but what matrices?

The numerator of the expression for x contains d and b just like the numerator, so the second column of the matrix we are seeking could be the same as that of the coefficient matrix. If we were to replace the first column of the coefficient matrix with the right hand side of the matrix form of the system we get

$$\begin{bmatrix} e & b \\ f & d \end{bmatrix},$$

and the determinant of this matrix is the numerator of the expression for x. Note that *the first column of the coefficient matrix is the coefficients of x*. Similarly, if we take the coefficient matrix and replace the *second* (y) column with the right hand side numbers we get

$$\begin{bmatrix} a & e \\ c & f \end{bmatrix},$$

and the determinant of this is $af - ce$, the numerator of the expression for y. Using the "bar notation" for the determinant, we now have

Cramer's Rule

The system of two equations in two unknowns with standard and matrix forms

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

has solution given by

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{and} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

We reiterate: the denominators of both of these fractions are the determinant of the coefficient matrix. The numerator for finding x is the determinant of the matrix obtained when the coefficient matrix has its first column (the coefficients of x) replaced with the numbers to the right of the equal signs. Let's see an example:

◇ **Example 7.5(a):** Use Cramer's rule to solve the system of equations $\begin{array}{rcl} 5x+3y & = & 1 \\ 4x+2y & = & 2 \end{array}$.

Solution: Cramer's Rule gives us

$$x = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix}} = \frac{1\cdot 2 - 2\cdot 3}{5\cdot 2 - 4\cdot 3} = \frac{-4}{-2} = 2 \qquad y = \frac{\begin{vmatrix} 5 & 1 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix}} = \frac{5\cdot 2 - 4\cdot 1}{5\cdot 2 - 4\cdot 3} = \frac{6}{-2} = -3$$

Section 7.5 Exercises To Solutions

1. Find the determinant of each of the following 2×2 matrices.

(a) $\begin{bmatrix} -2 & 5 \\ 4 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & -3 \\ 4 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 7 \\ 0 & 4 \end{bmatrix}$

2. Use Cramer's Rule to solve each of the following systems of equations.

(a) $\begin{array}{rcl} -2x+5y & = & 13 \\ 4x+7y & = & 25 \end{array}$ (b) $\begin{array}{rcl} 1x-3y & = & -17 \\ -2x+5y & = & 29 \end{array}$

(c) $\begin{array}{rcl} 8x-3y & = & 32 \\ 4x-5y & = & 16 \end{array}$ (d) $\begin{array}{rcl} 1x+7y & = & 48 \\ 4y & = & 28 \end{array}$

3. Solve the following systems using Cramer's Rule.

(a) $\begin{array}{rcl} x-3y & = & 6 \\ -2x+5y & = & -5 \end{array}$ (b) $\begin{array}{rcl} 2x-3y & = & -6 \\ 3x-y & = & 5 \end{array}$ (c) $\begin{array}{rcl} x+y & = & 3 \\ 2x+3y & = & -4 \end{array}$

(d) $\begin{array}{rcl} 7x-6y & = & 13 \\ 6x-5y & = & 11 \end{array}$ (e) $\begin{array}{rcl} 5x+3y & = & 7 \\ 3x-5y & = & -23 \end{array}$ (f)

$\begin{array}{rcl} 5x-3y & = & -11 \\ 7x+6y & = & -12 \end{array}$

4. Attempt to solve the following two systems of equations, using Cramer's Rule. What goes wrong?

(a) $\begin{array}{rcl} 2x-5y & = & 3 \\ -4x+10y & = & 1 \end{array}$ (b) $\begin{array}{rcl} 2x-5y & = & 3 \\ -4x+10y & = & -6 \end{array}$

A Solutions to Exercises

A.1 Chapter 1 Solutions

Section 1.1 Solutions Back to 1.1 Exercises

1. (a) $x = -\frac{1}{2}$ (b) $x = 10$ (c) $x = -5$ (d) $x = 5$ (e) $x = \frac{10}{11}$
   (f) $x = \frac{4}{3}$ (g) $x = \frac{4}{5}$ (h) $x = -15$
2. (a) $x = 12, 1$ (b) $x = 0, -1, -2$ (c) $x = 0, -\frac{1}{5}$ (d) $x = -5, \frac{3}{2}$
   (e) $a = -3, -5$ (f) $x = 0, 2$ (g) $x = -4, 4$ (h) $x = -\frac{1}{2}, 3$
   (i) $x = -2, 0, 11$
3. (a) $x = 1 + \sqrt{5},\ 1 - \sqrt{5}, \qquad x = -1.2, 3.2$
   (b) $x = 1 + 3\sqrt{2},\ 1 - 3\sqrt{2}, \qquad x = -3.2, 5.2$
   (c) $x = \frac{1}{2} + \frac{\sqrt{3}}{2},\ \frac{1}{2} - \frac{\sqrt{3}}{2}, \qquad x = -0.4, 1.4$
   (d) $x = -\frac{1}{5} + \frac{\sqrt{7}}{5},\ -\frac{1}{5} - \frac{\sqrt{7}}{5}, \qquad x = -0.7, 0.3$
4. (a) $x = -2, -\frac{2}{3}$ (b) $x = -\frac{2}{7}, \frac{4}{7}$ (c) $x = -4, 4, -\sqrt{10}, \sqrt{10}$
6. (a) The ball will be at a height of 84 feet at $t = 1$ seconds and $t = 5$ seconds.
   (b) The ball will be at a height of 148 feet at $t = 3$ seconds.
   (c) The ball will be at a height of 52 feet at $t = 0.55$ seconds and $t = 5.45$ seconds.
   (d) The ball never reaches a height of 200 feet.
   (e) The ball hits the ground at $t = 6.04$ seconds.
7. The price should be set at \$4.00 or \$8.00.
8. (a) $x = 0, \frac{4}{3}$ 11. (e) $x = -3, -2, 1$ 12. $x = 1, 2, 7$
13. (a) $x = -1,\ 2,\ 3$ (b) $x = -1,\ 1,\ 3,\ 4$ (c) $x = -2,\ 5$
18. (c) $x = -2, 2$
19. (a) $x = -\frac{6}{5}$ (b) $x = 3 + 2\sqrt{7},\ 3 - 2\sqrt{7}, \quad x = -2.3, 8.3$ (c) $x = -18$
    (d) $x = \frac{5}{3}, \frac{1}{3}$ (e) $x = \frac{2}{3} + \frac{\sqrt{5}}{3},\ \frac{2}{3} - \frac{\sqrt{5}}{3}, \quad x = -0.1, 1.4$ (f) $x = 0, 2, 5$
    (g) $x = 3, -7$ (h) $x = \frac{1}{4}$ (i) $x = \frac{1}{3}, -\frac{5}{2}$
    (j) $x = -4 + \sqrt{3},\ -4 - \sqrt{3}, \quad x = -2.3, -5.7$ (k) $x = -3, \frac{1}{2}$ (l) $x = -3, 7$

Section 1.2 Solutions Back to 1.2 Exercises

1. (a) $x = 4$ (b) $x = -\frac{11}{7}$ (c) $x = 2$
   (d) $x = 2$ (e) $x = 3$ (f) $x = 3$

2. (a) $x = \frac{4}{5}$ (b) $x = \frac{9}{2}$ (c) $x = 1$ ($x = -4$ is not a solution)
   (d) $x = -1$ ($x = -5$ is not a solution) (e) no solution ($x = \frac{2}{3}$ is not a solution)
   (f) $x = 2, 3$

3. (a) $x = 6$ (b) $x = \frac{7}{3}$ (c) $x = -1,\ 2$
   (d) $x = -4$ (e) $x = -\frac{1}{2},\ 5$ (f) no solution

6. (a) $x = -2, \frac{5}{2}$ (b) $x = -5, 6$ (c) $x = \frac{7}{2}, -\frac{3}{5}$ (d) $x = -5$
   (e) $x = \frac{8}{3}$ (f) $x = -5 + 2\sqrt{3},\ -5 - 2\sqrt{3},\quad x = -8.5, -1.5$
   (g) $x = 1, -\frac{1}{6}$ (h) $x = 8$ ($x = -1$ is not a solution)

Section 1.3 Solutions Back to 1.3 Exercises

1. The other leg has length 5 inches.
2. The radius is 2.6 inches.
3. (a) The sales tax is \$1.10.
   (b) You will pay \$ 21.05 with tax included.
4. In that month they make \$2870.50.
5. (a) \$1640 (b) \$3350
6. You borrowed \$1650.
7. The length is 40 and the width is 13.
8. The price of the item was \$189.45.
9. The legs of the triangle are 6.7 feet and 13.4 feet long.
10. Your hourly wage was \$6.70 per hour. (If you got \$6.63 your method is incorrect!)
11. The length is 11 units and the width is 4 units.
12. The price of the can opener is \$38.85.
13. The shortest side is 10.5 inches long, the middle side is 11.9 inches long, and the longest side is 21 inches long.
14. Her cost is \$42.82.
15. The lengths of the sides of the triangle are 8, 15 and 17 units long.
16. (a) The area of the rectangle is 70 square inches.
    (b) The new area is 280 square inches.
    (c) No, the new area is *FOUR* times as large.
    (d) The new area is 630 square inches, nine times larger than the original area.

17. (a) You will have \$1416.00 if you take it out after four years.
 (b) It will take 22.2 years to double your money.

18. $r = \dfrac{A-P}{Pt}$ or $r = \dfrac{A}{Pt} - \dfrac{1}{t}$ 19. $R = \dfrac{PV}{nT}$ 20. $b = \dfrac{ax+8}{x}$ or $b = a + \dfrac{8}{x}$

21. (a) 77°F (b) 35°C (c) $C = \dfrac{5F-160}{9}$ or $C = \frac{5}{9}F - 17\frac{7}{9}$ or $c = \frac{5}{9}(F-32)$
 (d) 25°C

22. $l = \dfrac{P-2w}{2}$ or $l = \frac{P}{2} - w$ 23. $r = \dfrac{C}{2\pi}$

24. (a) $y = -\frac{3}{4}x - 2$ (b) $y = -\frac{5}{2}x - 5$ (c) $y = \frac{3}{2}x + \frac{5}{2}$ (d) $y = \frac{3}{4}x - 2$
 (e) $y = -\frac{3}{2}x + \frac{5}{2}$ (f) $y = \frac{5}{3}x + 3$ 25. $x = -\frac{10}{3}$

26. (a) $x = \dfrac{d-b}{a-c}$ (b) $x = \dfrac{8.99}{2.065} = 4.35$ (c) $x = \dfrac{-8}{a-b}$ (d) $x = \dfrac{bc-7}{a-b}$
 27. $P = \dfrac{A}{1+rt}$

Section 1.4 Solutions Back to 1.4 Exercises

4. (a) 5 is the largest possible value of x, giving the solution $(5,-1)$.
 (b) $x = 4,\ 1,\ -4$

7. (a) x-intercept: 2 y-intercepts: 1, 4 (b) x-intercepts: $-4,\ 0$ y-intercept: 0

8. (a) x-intercept: 4 y-intercepts: $-2,\ 2$
 (b) x-intercepts: $-0.5,\ 4.5$ y-intercept: -1
 (c) x-intercept: -3 y-intercept: ≈ 1.8

9. x-intercept: 2.5 y-intercept: 5

10. x-intercept: -3 y-intercept: 5 11. x-intercepts: $3,\ -1$ y-intercept: -3

12. x-intercept: $\frac{9}{2}$ or $4\frac{1}{2}$ y-intercept: 3

13. (a) The x-intercepts are $(0,0)$ and $(5,0)$. (b) The y-intercept is $(0,0)$.

14. (a) x-intercepts: 0 y-intercepts: 0, 2 (b) x-intercepts: -4 y-intercepts: 2
 (c) x-intercepts: -3 y-intercepts: $\frac{3}{2}$ (d) x-intercepts: -3 y-intercepts: 9
 (e) x-intercepts: $1,-1$ y-intercepts: 1 (f) x-intercepts: 0 y-intercepts: 0
 (g) x-intercepts: 10 y-intercepts: 6 (h) x-intercepts: $3,-3$ y-intercepts: $4,-4$
 (i) x-intercepts: -3 y-intercepts: 1 (j) x-intercepts: none y-intercepts: 4
 (k) x-intercepts: -2 y-intercepts: 2

15. (a) The x-intercepts are $x = -3, 1, 2$ and the y-intercept is $y = 6$.

(b) The x-intercepts are $x = 5, -2, -4$ and the y-intercept is $y = 24$.

(c) The x-intercepts are $x = -2, 2$ and the y-intercept is $y = \frac{8}{3}$.

(d) The x-intercept is $x = 0$ and the y-intercept is $y = 0$.

Section 1.5 Solutions

Back to 1.5 Exercises

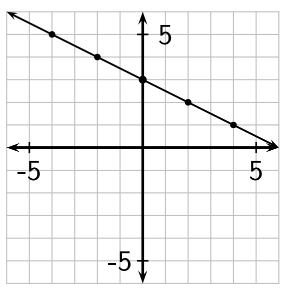
1. (a) $-\frac{2}{3}$ (b) $\frac{2}{5}$ (c) 0 (d) undefined (e) 3 (f) $-\frac{1}{2}$

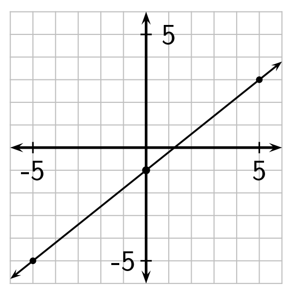
2. (a) $-\frac{5}{2}$ (b) 0 (c) $\frac{3}{4}$ (d) undefined (e) $\frac{4}{3}$ (f) $-\frac{1}{3}$

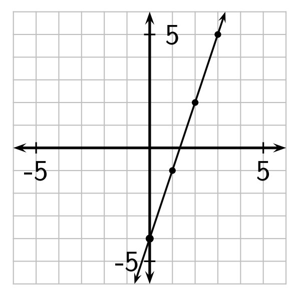
3. (a) $y = -\frac{5}{2}x + 3$ (b) $y = 3$ (c) $y = \frac{3}{4}x - 2$
   (d) $x = 3$ (e) $y = \frac{4}{3}x$ (f) $y = -\frac{1}{3}x + 2$

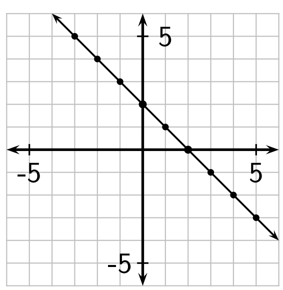
4. 3

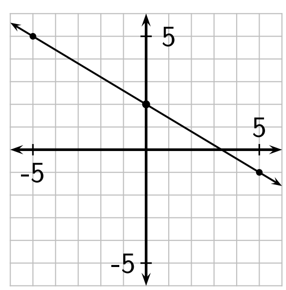
5. $\frac{1}{2}$

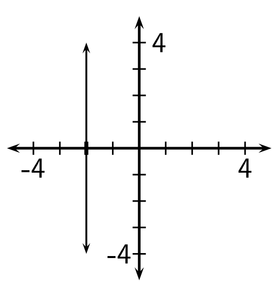
6. (a)


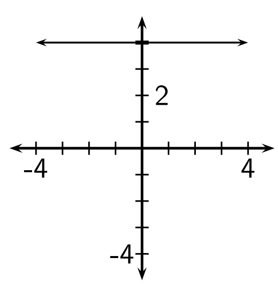
(b)


(c)


(d)


(e)


7. (a)


(b)


8. (a) $y = \frac{4}{3}x + 2$ (b) $y = -\frac{2}{3}x + 7$ (c) $y = 2$
   (d) $y = \frac{1}{4}x - 5$ (e) $y = -\frac{3}{4}x + \frac{1}{2}$ (f) $y = 4$

9. (a) $m = \frac{5}{3}$ (b) $m = \frac{5}{3}$ (c) $m = -\frac{3}{5}$

10. $3x - 4y = 7 \;\Rightarrow\; y = \frac{3}{4}x - \frac{7}{4}$, $8x - 6y = 1 \;\Rightarrow y = \frac{4}{3}x - \frac{1}{6}$ The lines are neither parallel nor perpendicular.

11. $y = \frac{3}{2}x - 2$ 12. $y = \frac{3}{4}x + 3$ 13. $y = -5$ 14. $y = -\frac{4}{5}x - \frac{7}{5}$

15. $y = -\frac{2}{3}x + 6$ 16. $y = -5x + 10$ 17. $y = \frac{4}{3}x - 2$

18. The slope of the line through A and B is $\frac{3}{4}$ and the slope of the line through B and C is $\frac{6}{8} = \frac{3}{4}$. The three points lie on the same line.

19. $y = \frac{7}{5}x + 9$

Section 1.6 Solutions

Back to 1.6 Exercises

1. (a) $(3, 7)$ (b) $(3, 4)$ (c) $(13, -10)$

2. (a) $(1, -1)$ (b) $(-1, 4)$ (c) $(-2, \frac{1}{3})$

6. $(\frac{29}{22}, -\frac{5}{11})$ 7. $(8, 6)$

8. The plane's speed in still air is 540 mph and the wind speed is 60 mph.

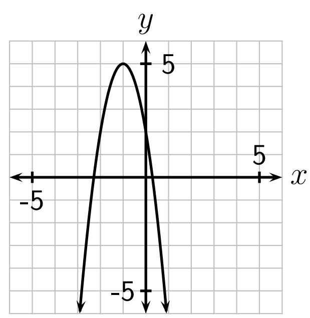
9. There are 5 people ahead of you and 3 people behind you.

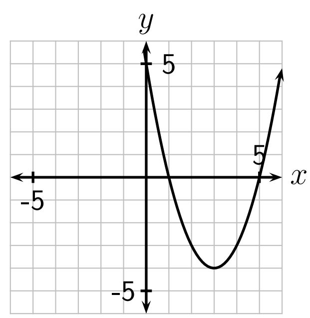
10. 3.9 million bushels are produced and sold at equilibrium.

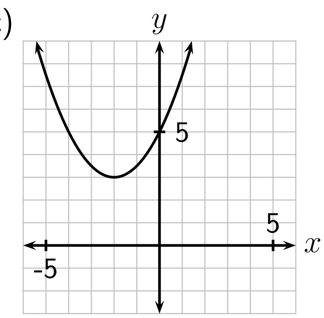
Section 1.7 Solutions

Back to 1.7 Exercises

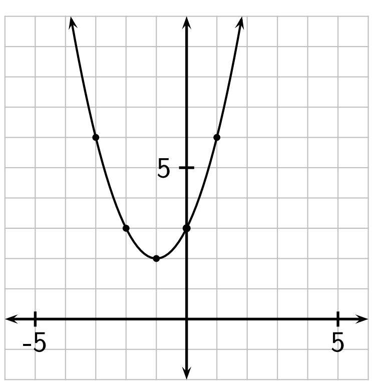
1. (a) downward, $(-1, 5)$ (b) upward, $(3, -4)$ (c) upward, (-2,3)

2. (a)


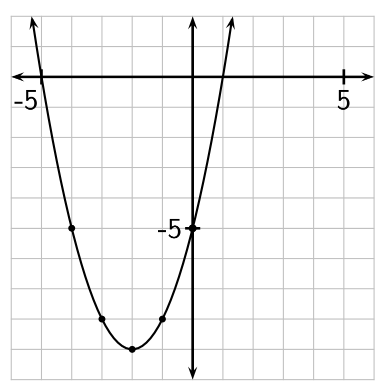
(b)


(c)


3. See graph below and to the left.



Exercise 3



Exercise 4

4. See graph above and to the right. The graph has the same x-intercepts as $y = -x^2 - 4x + 5$,

but the the y-intercept and y-coordinate of the vertex are the opposites of what they are for $y = x^2 + 4x - 5$.

5. The graph of $y = ax^2 + bx + c$ can have 0, 1 or 2 x-intercepts.

6. (a) 6.5 (b) $(1, 4)$

   (c) The other x-intercept is 1 and the point $(-4, 5)$ is on the parabola.

7. (a) upward (b) -7 (c) $-1,\ 7$

   (d) See below. The x-coordinate of the vertex is 3.

   (e) The y-coordinate of the vertex is -16. (f) See with Exercise 9 solutions.

   (d) The minimum value of the function is -16, which occurs at $x = 3$.

8. (a) The x-intercept cannot be found. When we try to use the quadratic formula to find the x-intercepts, we get a negative number under a square root.

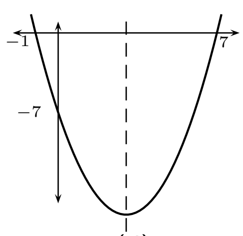
   (b) The parabola opens upward.

   (c) The vertex is $(3, 4)$.

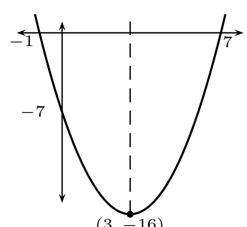
   (d) Because the vertex is above the x-axis and the parabola opens upward, there are no x-intercepts.

   (e) $(2, 5),\ (4, 5),\ (1, 8),\ (5, 8),\ (0, 13),\ (6, 13)$, etc.

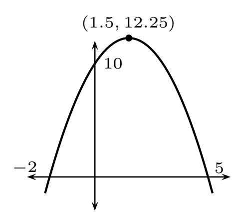
9. (a) $a = -1,\ b = 3,\ c = 10$ (b) See below.



7(d)

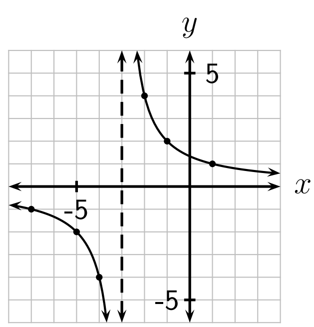


7(f)

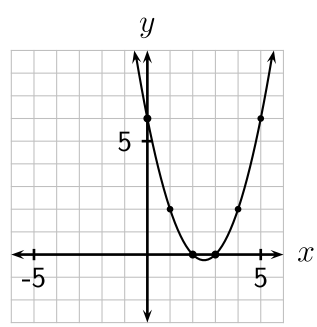


9(b)

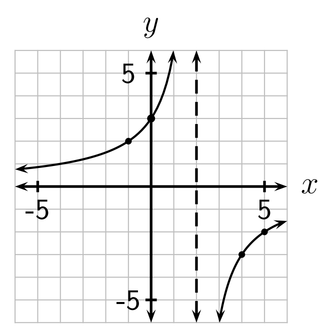
10. (a)



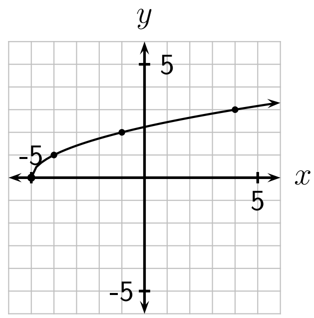
(b)



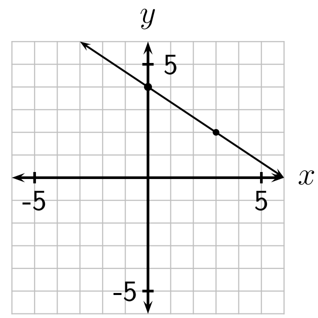
(c)



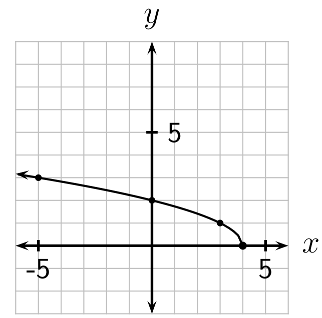
(d)



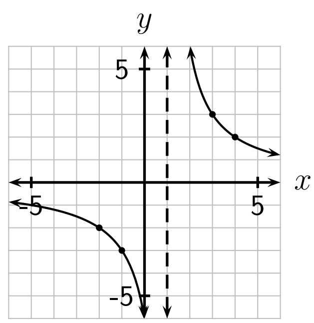
(e)



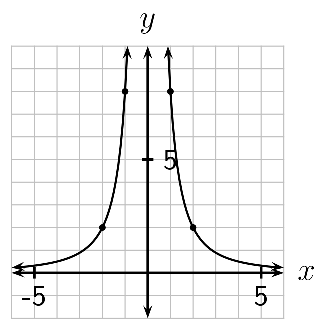
(f)



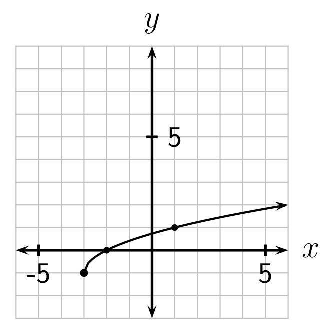
(g)



(h)



(i)



Section 1.8 Solutions Back to 1.8 Exercises

1. (a) $x = -16$ (b) $x = 5$ (c) $x = 1, 9$ (d) $x = 3+\sqrt{2},\ 3-\sqrt{2},\ \ x = 1.4, 4.4$
   (e) $x = 0, \frac{4}{3}$ (f) no solution ($x = -4$ is not a solution) (g) $x = \frac{7}{3}$
   (h) $y = -4, 2$ (i) $x = 4$ (j) $x = -6, 6$

4. The length is 18.0 inches and the width is 10.6 inches.

5. The length is 11 units and the width is 8.5 units.

6. The price of the compact disc is \$9.95.

7. (a) $y = -\frac{5}{2}x + 10$ (b) $y = \frac{2}{3}x - 2$ (c) $y = -\frac{3}{5}x - 2$

8. (a) $x = \dfrac{c}{a+b}$ (b) $x = \dfrac{-10}{a-c}$

9. (a) The projectile will be at a height of 20 feet at $t = \frac{1}{2}$ seconds and $t = 2\frac{1}{2}$ seconds.
   (b) The projectile hits the ground after 3 seconds.
   (c) At 1.92 seconds the projectile will be at a height of 33.2 feet.
   (d) The projectile will be at a height of 32 feet at $t = 1$ seconds and $t = 2$ seconds.
   (e) The projectile will be at a height of 36 feet at $t = 1\frac{1}{2}$ seconds. This is different in that it is at that height at only one time, because 36 feet is the highest point.
   (f) The projectile never reaches a height of 40 feet.

10. The price should be set at \$6.50.

11. (a) x-intercepts: 0, 2 y-intercepts: 0 (b) x-intercepts: 4 y-intercepts: 2

(c) x-intercepts: 1 y-intercepts: none (d) x-intercepts: $5, -5$ y-intercepts: $5, -5$

(e) x-intercepts: 5 y-intercepts: -4 (f) x-intercepts: 2 y-intercepts: -4

(g) x-intercepts: $-\frac{3}{2}, \frac{3}{2}$ y-intercepts: -3

12. (a) $y = 3x + 10$ (b) $x = 1$ (c) $y = 4x - 1$

(d) $y = 2x + 5$ (e) $y = -\frac{1}{11}x - \frac{28}{11}$ (f) $y = \frac{5}{2}x$

13. (a) $-2y + 7z = 16$ (b) $7y + 3z = -1$ (c) $z = 2$ (d) $y = -1$ (e) $x = 3$

14. (a) $(2, 0, 3)$ (b) $(1, \frac{1}{2}, \frac{9}{2})$ (c) $(4, 3, 2)$ (d) $(-2, 1, 4)$

15. downward, $(\frac{3}{2}, \frac{49}{4})$

16. (a) Parabola opening downward, x-intercepts $(-3, 0)$, $(5, 0)$, y-intercept $(0, 15)$, vertex $(1, 16)$. See graph below. The function has an absolute maximum of 16 at $x = 1$.

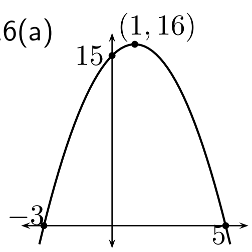
(b) Parabola opening upward, x-intercepts $(2, 0)$, $(7, 0)$, y-intercept $(0, 14)$, vertex $(4\frac{1}{2}, -6\frac{1}{4})$. See graph below. The function has an absolute minimum of $-6\frac{1}{4}$ at $x = 4\frac{1}{2}$.

(c) Parabola opening downward, x-intercepts $(1, 0)$, $(9, 0)$, y-intercept $(0, -2\frac{1}{4})$, vertex $(5, 4)$. See graph below. The function has an absolute maximum of 5 at $x = 4$.

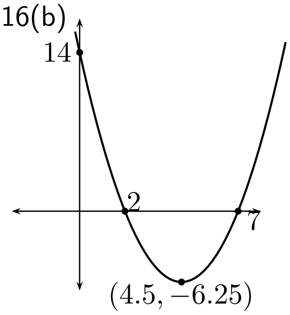
(d) Parabola opening upward, x-intercept $(-3, 0)$, y-intercept $(0, 4\frac{1}{2})$, vertex $(-3, 0)$. See graph below. The function has an absolute minimum of 0 at $x = -3$.

19. $y = \frac{7}{5}x + 9$

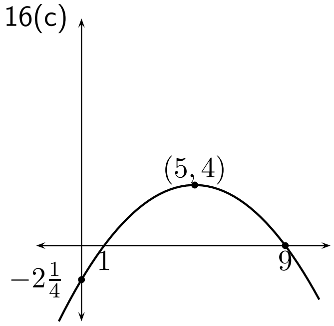
16(a)



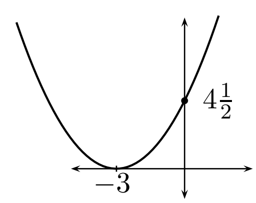
16(b)



16(c)



16(d)

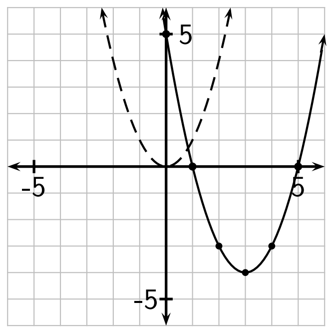


17. (a) $f(x) = x^2 - 6x + 5$

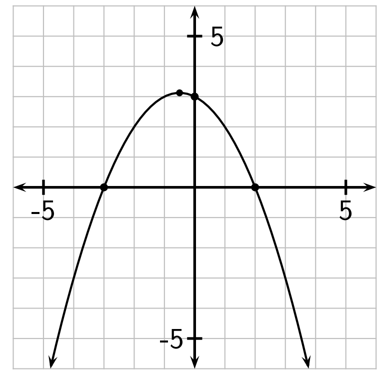
(b) $(3, -4)$ The x-coordinate is the value that makes $(x - 3)$ zero, and the y-coordinate is the -4 at the end of the equation.

(c) The x-intercepts are 1 and 5, and the y-intercept is 5.

(d) See on the next page.



Exercise 17(d)



Exercise 18(e)

18. (a) a would be $-\frac{1}{2}$, so the parabola opens down.
(b) The y-intercept is $y = -\frac{1}{2}(0-2)(0+3) = 3.$
(c) The x-intercepts are 2 and $-3.$
(d) The x-coordinate of the vertex is $x = \dfrac{2+(-3)}{2} = -\dfrac{1}{2}.$
The y-coordinate of the vertex is $y = -\frac{1}{2}(-\frac{1}{2}-2)(-\frac{1}{2}+3) = -\frac{1}{2}(-\frac{5}{2})(\frac{5}{2}) = \frac{25}{8} = 3\frac{1}{8}.$
(e) See above.

A.2 Chapter 2 Solutions

Section 2.1 Solutions Back to 2.1 Exercises

1. (a) $h(12) = 17$ (b) $x = \frac{19}{2}$

2. (a) $f(3) = 9,\ f(-5) = 65$ (b) $x = -\frac{3}{2},\ 3$ (c) $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$ or $x = -0.28,\ 1.78$

3. (a) $x = 1, 6$ (b) $g(-2) = -3$ (c) $x = -0.35,\ 3.95$

4. (a) $f(6) = \frac{11}{2}$ (b) $x = \frac{16}{3}$

5. (a) (a) $h(-4) = 3$ (b) $h(2) \approx 4.58$
   (c) $h(7)$ doesn't exist, because we can't take the square root of a negative number.
   (d) We must make sure that $25 - x^2 \geq 0$. This holds for all values of x between -5 and 5, including those two.

6. $h(0) = h(-43) = h(3\frac{4}{7}) = 12$

7. (a) x-intercept: $x = \frac{7}{2}$ y-intercept: $y = -7$
   (b) x-intercepts: $x = 0, \frac{3}{2}$ y-intercept: $y = 0$
   (c) x-intercept: $x = \frac{1}{2} + \frac{\sqrt{13}}{2},\ \frac{1}{2} - \frac{\sqrt{13}}{2}$ y-intercept: $y = -\sqrt{3}$
   (d) x-intercept: $x = -\frac{4}{3}$ y-intercept: $y = -2$
   (e) x-intercept: $x = -5, 5$ y-intercept: $y = 5$
   (f) x-intercept: there isn't one y-intercept: $y = 12$

8. (a) $f(x+1) = x^2 - x - 2$ (b) $g(2x) = \sqrt{2x+1}$
   (c) $f(-2x) = 4x^2 + 6x$ (d) $g(x-4) = \sqrt{x-3}$

9. (a) $g(a-3) = a^2 - a - 6$ (b) $g(x+h) = x^2 + 2xh + h^2 + 5x + 5h$
   (c) $\dfrac{g(x+h) - g(x)}{(x+h) - x} = 2x + h + 5$

10. (a) The x-intercepts are $x = -7,\ 1,\ 3,\ x \approx -2.5,$ and the y-intercept is $y \approx -1.5$
    (b) $f(-3) = 1,\ f(-2) = -1,\ f(3) = 0$ (c) $x = -6, -3.5$ (d) $x = -5$

11. (a) $f(-3) = 1,\ f(-1) = -2,\ f(1) = 4$ (b) $x \approx -6.6,\ -3,\ 0.3,\ 5.9$ (c) $f(0) = 0$
    (d) $x = -5,\ 1,\ 1.1,\ \approx 5.5$ (e) $x \approx 2.7,\ 4.3$

12. (a) $f(4) = g(4) = 4$ (c) $x = 2$

13. $t = 2$ (-2 doesn't check) 14. $y = 3$ (1 doesn't check)

18. $f(x+h) = x^2 + 2xh + h^2 - 5x - 5h + 2$ 19. $g(x+h) = 3x + 3h - x^2 - 2xh - h^2$

Section 2.2 Solutions Back to 2.2 Exercises

1. (a) $\{x \mid 2\frac{1}{2} \leq x \leq 13\}$ (b) $\{x \mid -12 < x \leq -2\}$

(c) $\{x \mid -100 < x < 100\}$ (d) $\{x \mid x > 3\}$ (e) $\{x \mid x \leq -5\}$

2. (a) $[2\frac{1}{2}, 13]$ (b) $(-12, -2]$ (c) $(-100, 100)$ (d) $(3, \infty)$ (e) $(-\infty, -5]$

3. (a) $(-\infty, 3) \cup [7, \infty)$ (b) $(-\infty, -4) \cup (4, \infty)$ (c) $(-\infty, 7) \cup (7, \infty)$

4. (a) $\{x \mid x < 3 \text{ or } x \geq 7\}$ (b) $\{x \mid x < -4 \text{ or } x > 4\}$ (c) $\{x \mid x \neq 7\}$

5. (a) $[-2, \infty)$ or $\{x \mid x \geq -2\}$ (b) $\{x \mid x \neq 1\}$ (c) $\{x \mid x \neq -5, 2, \}$

6. (a) The domain is $[-3, \infty)$ and the range is $[-1, \infty)$
   (b) The domain is $\{x \mid x \neq 2\}$ and the range is $\{y \mid y \neq 0\}$
   (c) The domain is all real numbers and the range is $(-2, \infty)$
   (d) The domain is all real numbers and the range is $(0, 1]$

7. (a) $[4, \infty)$ $\{x \mid x \geq 4\}$ (b) $(-\infty, 4]$ or $\{x \mid x \leq 4\}$ (c) $\{x \mid x \neq 3\}$
   (d) $[-5, 5]$ or $\{x \mid -5 \leq x \leq 5\}$ (e) $\{x \mid x \neq -5\}$ (f) $\{x \mid x \neq \frac{2}{3}\}$
   (g) $\{x \mid x \neq 1, 4\}$ (h) $[1, 5) \cup (5, \infty)$ or $\{x \mid 1 \leq x$ and $x \neq 5\}$ (i) $\mathbb{R}$ or all real numbers

8. (a) $\{x \mid x \geq 3\}$, $[3, \infty)$ (b) $\{x \mid x \leq 3\}$, $(-\infty, 3]$
   (c) The domain of h does not include 3, because $x = 3$ would cause a zero in the denominator. The domain of h is $\{x \mid x < 3\}$, $(-\infty, 3)$

9. (a) all real numbers (b) $(-\infty, 5]$ (c) $[-9, \infty)$

Section 2.3 Solutions Back to 2.3 Exercises

1. (a) The x-intercepts are $x = -7, 1, 3,$ $x \approx -2.5$. The y-intercept is $y \approx -1.5$.
   (b) The domain of f is all real numbers.
   (c) The range of f is $(-\infty, 3]$
   (d) f is increasing on the intervals $(-\infty, -5)$ and $(-1, 2)$.
   (e) f is negative on $(-\infty, -7)$, $(-2.5, 1)$ and $(3, \infty)$.
   (f) There is a relative maximum of 3 at $x = -5$, a relative maximum of 1 at $x = 2$ and a relative minimum of -2 at $x = -1$.
   (g) f has an absolute maximum of 3 at $x = -5$.
   (h) f has no absolute minimum.
   (i) $f(-4) \approx 2.5$
   (j) $f(x) = -1$ at $x = -2$ and $x \approx 0.4$

2. (a) The domain of g is all real numbers. (b) The range of g is $(-3, \infty)$.
   (c) $g(-2) \approx -2$ and $g(-3) \approx -2.5$. (d) $g(x) = -1$ when $x = -1$.
   (e) $g(x) = 5$ when $x = 1$.
   (f) g is increasing on the interval $(-\infty, \infty)$.

(g) g does not have any relative (or absolute) maxima or minima.

(h) g is positive on about $(-0.4, \infty)$.

3. (a) $f(-x) = -3x + 1$, the function is neither even nor odd

   (b) $f(-x) = x^2 - 5 = f(x)$, the function is even

   (c) $f(-x) = -5x^3 + 7x + 1$, the function is neither even nor odd

   (d) $f(-x) = 10 = f(x)$, the function is even

   (e) $f(x) = -x^5 - 4x = -f(x)$, the function is odd

4. (a) odd (b) neither (c) odd

   (d) even (e) neither (f) even

5. Every quadratic function has exactly one absolute maximum (if the graph of the function opens downward) or one absolute minimum (if the graph opens upward), but not both. The are no relative maxima or minima other than the absolute maximum or minimum.

Section 2.4 Solutions Back to 2.4 Exercises

1. (a) 17.4 mpg, 17.3 mpg, The mileage decreased by 0.1 mpg as the speed increased from 45 to 50 mph.

   (b) 16.5 mpg, 16.4 mpg, The mileage decreased by 0.1 mpg as the speed increased from 90 to 95 mph.

   (c) The slope of the line is -0.02, and when you multiply it by 5 you get -0.1. This the change in mileage (negative indicating a decrease) in mileage as the speed is increased by 5 mph.

   (d) 40 mph

   (e) The slope of the line is $-0.02\ \frac{\text{mpg}}{\text{mph}}$. This tells us that the car will get 0.02 miles per gallon less fuel economy for each mile per hour that the speed increases.

   (f) We should not interpret the value of the m-intercept because it should represent the mileage when the speed is zero miles per hour. We are told in the problem that the equation only applies for speeds between 30 and 100 miles per hour. Furthermore, when the speed is zero mph the mileage should be infinite, since no gas is being used!

2. (a) The weight of a lizard that is 3 cm long is -18 grams. This is not reasonable because a weight cannot be negative. The problem is that the equation is only valid for lizards between 12 and 30 cm in length.

   (b) The w-intercept is -84, which would be the weight of a lizard with a length of zero inches. It is not meaningful for the same reason as given in (a).

   (c) The slope is 22 grams per centimeter. This says that for each centimeter of length gained by a lizard, the weight gain will be 22 grams.

3. (a) $2300, $3800 (b) $P = 0.03S + 800$

   (c) the slope of the line is 0.03 dollars of pay per dollar of sales, the commission rate.

   (a) The P-intercept of the line is $800, the monthly base salary.

4. (a) The F-intercept is 32°, which is the Fahrenheit temperature when the Celsius temperature is 0 degrees.
   (b) The slope of the line is $\frac{9}{5}$ degrees Fahrenheit per degree Celsius; it tells us that each degree increase in Celsius temperature is equivalent to $\frac{9}{5}$ degrees increase in Fahrenheit temperature.

5. (a) The C-intercept is $5000. This represents costs that Acme has regardless of how many Widgets they make. These are usually referred to as *fixed costs*.
   (b) The slope of the line is $7 per Widget. It says that each additional Widget produced adds $7 to the total cost. Business people often refer to this as "marginal cost."
   (c) The total cost is $15,437, the cost per Widget is $10.35

6. The slope of $0.24 per mile is the additional cost for each additional mile driven. The intercept of $30 is the the base charge for one day, without mileage.

7. (a) $W = 6t + 8$
   (b) The slope is 6 pounds per year, and represents the additional weight the baby will gain each year after birth.
   (c) The W-intercept of 8 pounds is the weight at birth, which is time zero.
   (d) The child weighed 26 pounds at 3 years of age.

8. $x = -600p + 9500$ 9. $T = \frac{2}{9}x + 40$

Section 2.5 Solutions Back to 2.5 Exercises

1. (a) 288 feet (b) 9 seconds (c) 324 feet
   (d) 2.26 seconds and 6.74 seconds. There are two times because the is at a height of 224 feet once on the way up, and again on the way down.

2. (a) $h(4)$ represents the height of the rock at four seconds after it was thrown.
   (b) $t = 1.20$ seconds, $t = 7.80$ seconds
   (c) The rock reaches a height of 150 feet at 1.20 seconds and 7.80 seconds after it was thrown.
   (d) Since the rock hits the ground at $t = 9$ seconds, it makes no sense to use the equation for $t = 11$ seconds.

3. (a) $k = 0.048$ (b) $\frac{\text{ft}}{\text{mph}^2}$ (c) $L = 120$ feet (d) $s = 45.6$ miles per hour

4. (a) At 18 feet the intensity should be *less* than 180 candlepower.
   (b) At 18 feet the intensity is 80 candlepower.

5. (a) 15000 can be sold at $50 and 10000 can be sold at $100; the number sold decreases by 5000.
   (b) The number sold decreases by 5000 again.
   (c) $130 (d) The feasible domain is $[0, 200]$.

6. (a) At a price of \$80, $w = 20,000 - 8000 = 12,000$ widgets will be sold. The revenues will then be $(12,000)(80) = \$960,000$
   (b) \$990,000, \$840,000
   (c) As the price increases from \$80 to \$110, the revenue increases by \$30,000. As the price increases from \$110 to \$140, the revenue *decreases* by \$150,000.
   (d) $R = wp = (20,000 - 100p)p = 20000p - 100p^2$
   (f) \$5.13 or \$194.87

7. (a) 3190 (b) You can't have part of a (live) fish! (c) 1714 (d) 0 or 3300

8. There will be 10.3 milligrams left after 3 hours.

9. (c) $d = 224 - 3.5t$ (d) 154 feet (e) $t \approx 26.9$ seconds, $t = 64$ seconds

10. (b) $P = (3 + 4t) + (5 + 4t) + \sqrt{(3 + 4t)^2 + (5 + 4t)^2}$
    (c) $P = 8 + 8t + \sqrt{32t^2 + 64t + 34}$

11. $A = 500x - 2x^2$

12. The mathematical domain is all real numbers, but the feasible domain is $(0, 250)$ because if x is over 250 there will not be enough fence to make the four sections with length x.

13. (a) Billy can mow 1.5 lawns in an hour and Bobby can mow 1 lawn in an hour.
    (b) $L = 1.5t + 1t = 2.5t$
    (c) Together the two of them can mow one lawn in $\frac{2}{5}$ of an hour, which is $\frac{2}{5}(60) = 24$ minutes.
    (d) The mathematical domain of the function is all real numbers, but the feasible domain is $[0, \infty)$.

14. (a) The three lengths/distance of interest are the length of the ladder, the height of the top of the ladder, and the distance of the base of the ladder from the wall. The height of the ladder and the distance of the base from the wall are variables, and the length of the ladder is a constant.
    (b) As the distance of the base from the wall decreases, the height of the top of the ladder increases.
    (c) $d^2 + h^2 = 10^2$ or $d^2 + h^2 = 100$
    (d) The height of the ladder is 8 feet when the base is 6 feet from the wall. When the base of the ladder is 2 feet from the wall the height is 9.8 feet.
    (e) 9.8 feet, $4\sqrt{6}$ feet.
    (f) The first value is nice because it is clear how much it is. The second number is hard for us to interpret, but it is exact. Someone could use it to get the answer to as many decimal places as they desired.
    (g) 9.8 feet rounded to the nearest inch is 118 inches, or 9 feet 10 inches.

15. (a) We can disregard the negative sign because the height must be a positive number.

(b) The feasible domain is $(0, 10)$.

16. (a) $A = l(\frac{1}{3}l) = \frac{1}{3}l^2$ (b) $A = w(3w) = 3w^2$

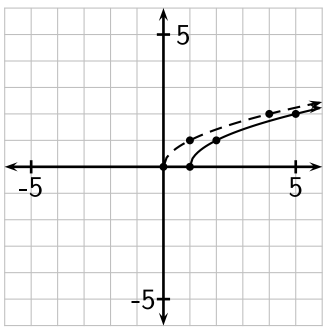
(c) $P = 2l + 2(\frac{1}{3}l) = \frac{8}{3}l$, $P = 2(3w) + 2w = 8w$

17. (a) $5000 + 7(8000) = \$61,000$, $5000 + 7(12000) = \$89,000$

(b) $C = 5000 + 7x$

Section 2.6 Solutions Back to 2.6 Exercises

2. (a)



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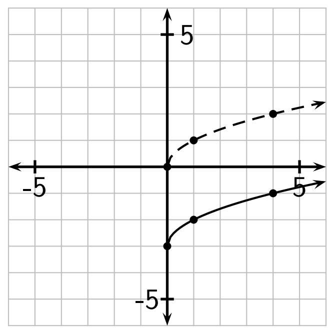

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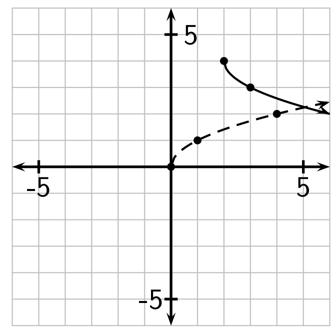
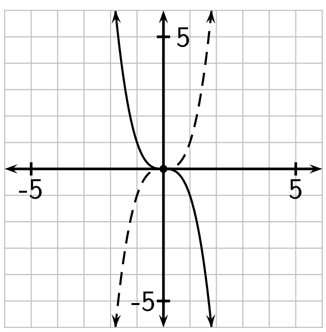


4. (a)



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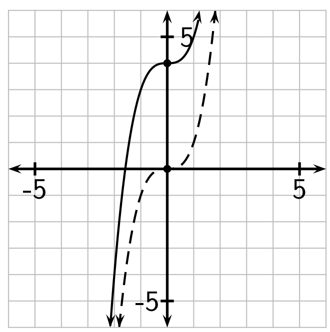

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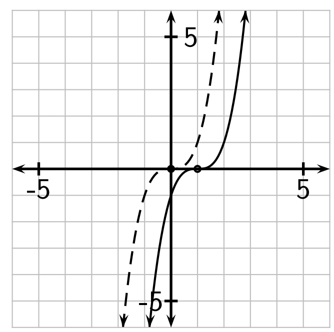

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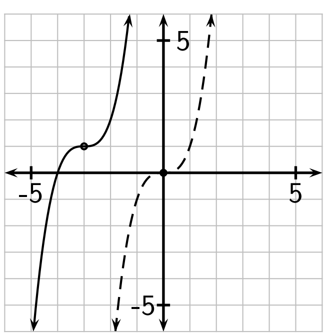
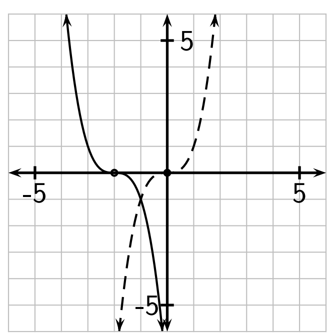
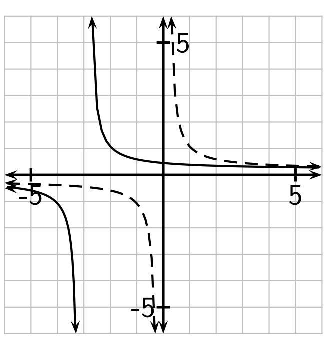


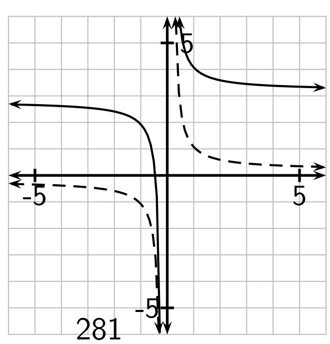
(e)



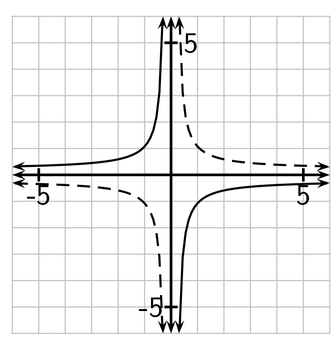
5. (a)

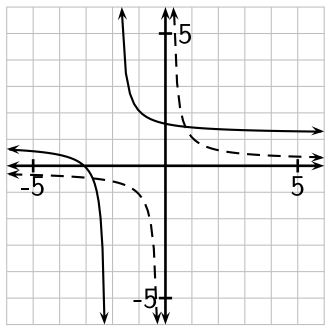


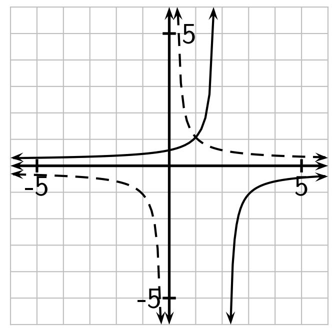
(b)



(c)



(d) 

(e) 

6. (a) Domain: $\mathbb{R}$ Range: $[-2, \infty)$
   (b) Dom$(f) = [-3, \infty)$ Ran$(f) = [0, \infty)$
   (c) Dom$(g) = \mathbb{R}$ Ran$(g) = (-\infty, 5]$
   (d) Dom$(h) = \{x \mid x \neq 2\}$ Ran$(h) = \{y \mid y \neq 0\}$
   (e) Domain: $\mathbb{R}$ Range: $(-\infty, 7]$
   (f) Domain: $[2, \infty)$ Range: $[1, \infty)$

Section 2.7 Solutions

Back to 2.7 Exercises

1. (a) $\{x \mid x \neq 0, 1\}$ (b) $[-2, \infty)$ or $\{x \mid x \geq -2\}$ (c) $\{x \mid x \neq -1, 1\}$
   (d) $\{x \mid x \neq 1, 4\}$ (e) $[\frac{2}{3}, \infty)$ (f) $\mathbb{R}$ or all real numbers

2. (a) Dom$(h) = \{x \mid x \neq -3, 2\}$ (b) Ran(h) is all real numbers.
   (c) $h(-4) = 3$ and $h(4) \approx 1$ (d) $h(x) = 1$ for $x = -1, 1, 4$
   (e) $h(x) = 2$ when $x = 0$ and $x = 3$.
   (f) h is *decreasing* on the intervals $(0, 2)$ and $(2, 4)$.
   (g) h is negative on $(-3, -1.9)$ (or $(-3, 2)$) and $(1.3, 2)$
   (h) h has a relative maximum of 2 at $x = 0$ and a relative minimum of 1 at $x = 4$.
   (i) h has no absolute maxima or minima.

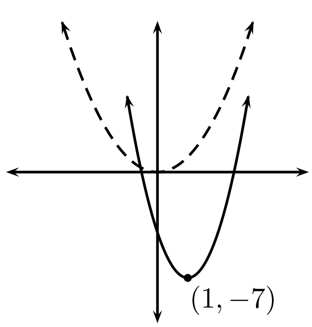
A.3 Chapter 3 Solutions

Section 3.1 Solutions Back to 3.1 Exercises

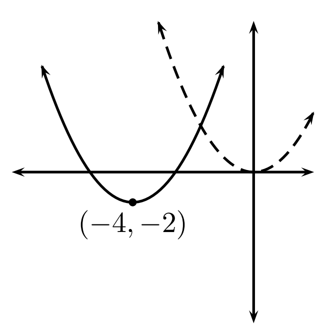
1. (a) $f(x) = -\frac{1}{5}x^2 - \frac{1}{5}x + \frac{6}{5}$ (b) $g(x) = 3x^2 - 6x + 5$
   (c) $y = -2x^2 + 18$ (d) $h(x) = \frac{1}{2}x^2 + 2x - 3$
2. (a) $f(x) = -\frac{1}{4}(x+1)(x-5)$ (b) $y = 2(x-3)(x-1)$
   (c) $g(x) = \frac{1}{2}(x+1)(x+9)$ (d) $h(x) = -5(x+1)(x-1)$
   (e) $y = 3(x-2)(x-3)$ (f) $f(x) = \frac{1}{4}(x+8)(x-2)$
   (g) $y = -\frac{1}{3}(x+2)(x-3)$ (h) $g(x) = -(x+2)(x-7)$
3. (a) $y = 5(x+1)(x-1)$ (b) $h(x) = x(x-4)$
4. (a) The x-intercepts are -1 and 5, the y-intercept is $\frac{5}{4}$ and the vertex is $(2, \frac{9}{4})$.
   (b) The x-intercepts are 1 and 3, the y-intercept is 6 and the vertex is $(2, -2)$.
   (c) The x-intercepts are -1 and -9, the y-intercept is $\frac{9}{2}$ and the vertex is $(-5, -8)$.
   (d) The x-intercepts are -1 and 1, the y-intercept is 5 and the vertex is $(0, 5)$.
   (e) The x-intercepts are -2 and 7, the y-intercept is 14 and the vertex is $(2\frac{1}{2}, 20\frac{1}{4})$.
   (f) The x-intercepts are -8 and 2, the y-intercept is -4 and the vertex is $(-3, -6\frac{1}{4})$.
5. (a) $f(x) = -x^2 - 2x + 15$ (b) $f(x) = -(x-3)(x+5)$ (c) $(-1, 16)$
6. (a) $y = \frac{1}{3}x^2 - \frac{4}{3}x + 1$ (b) $y = \frac{1}{3}(x-3)(x-1)$ (c) $(2, -\frac{1}{3})$
9. The graph of a quadratic function can have 0, 1 or 2 x-intercepts.
10. Every quadratic function has exactly one absolute maximum (if the graph of the function opens downward) or one absolute minimum (if the graph opens upward), but not both. The are no relative maxima or minima other than the absolute maximum or minimum.

Section 3.2 Solutions Back to 3.2 Exercises

1. (a)



(b)



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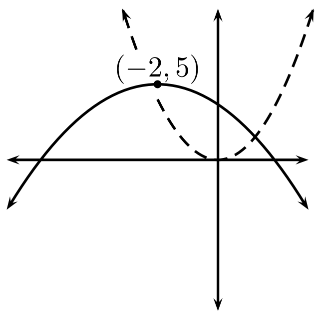
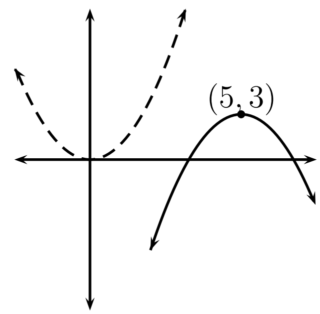


(d)



2. (a) $y = -\frac{9}{2}(x+5)^2 - 7$ (b) $y = \frac{2}{3}(x-6)^2 + 5$

   (c) $y = \frac{1}{9}(x-4)^2 - 3$ (d) $y = 5(x+1)^2 - 4$

4. $y = 3(x-1)^2 + 2$

5. $h(x) = -\frac{1}{4}(x-5)^2 + 4$

7. $y = -\frac{1}{2}(x-4)^2 + 1$

8. $y = -3(x-1)^2 - 7$

9. $y = \frac{1}{2}(x-3)^2 + 2$

10. $y = 5(x+2)^2 - 3$

Section 3.3 Solutions Back to 3.3 Exercises

1. (a) $x \geq \frac{14}{5}$ (b) $x \geq -3$ (c) $x \geq -1$

   (d) $x > -3$ (e) $x > 9$ (f) $y \geq -1$

2. (a) $1 < x < 8$, OR $(1, 8)$

   (b) $-2 \leq x \leq \frac{5}{2}$, OR $[-2, \frac{5}{2}]$

   (c) $-5 \leq x \leq \frac{3}{2}$, OR $[-5, \frac{3}{2}]$

   (d) $x < -\frac{1}{5}$ or $x > 0$, OR $(-\infty, -\frac{1}{5}) \cup (0, \infty)$

   (e) $-5 < x < 3$, OR $(-5, 3)$

   (f) $x \leq -6$ or $x \geq -1$, OR $(-\infty, -6] \cup [-1, \infty)$

3. (a) $x < -4$ or $-1 < x < 3$, OR $(-\infty, -4) \cup (-1, 3)$

   (b) $-2 \leq x \leq 2$ or $x \geq 5$, OR $[-2, -2] \cup [5, \infty)$

   (c) $-2 < x < 3$ or $x > 3$, OR $(-2, 3) \cup (3, \infty)$

   (d) $x \leq 2$, OR $(-\infty, 2]$

4. (a) $-2 \leq x \leq -1$ or $x \geq 0$, OR $[-2, -1] \cup [0, \infty)$

   (b) $x < -\frac{3}{5}$ or $0 < x < \frac{7}{2}$, OR $(-\infty, -\frac{3}{5}) \cup (0, \frac{7}{2})$

Section 3.4 Solutions

Back to 3.4 Exercises

1. (a) 34.6 feet
   (b) The rock will be at a height of 32 feet at 1 second (on the way up) and 2 seconds (on the way down).
   (c) 0.6 seconds, 2.4 seconds
   (d) The rock hits the ground at 3 seconds.
   (e) The mathematical domain is all real numbers. The feasible domain is [0,3]
   (f) The rock reaches a maximum height of 36 feet at 1.5 seconds.
   (g) The rock is at a height of less than 15 feet on the time intervals $[0, 0.35]$ and $[2.65, 3]$.

2. We can see that $f(x) = x - x^2$, so its graph is a parabola opening downward. From the form $f(x) = x(1-x)$ it is easy to see that the function has x-intercepts zero and one, so the x-coordinate of the vertex is $x = \frac{1}{2}$. Because $f(\frac{1}{2}) = \frac{1}{4}$, the maximum value of the function is $\frac{1}{4}$ at $x = \frac{1}{2}$.

3. $t = \dfrac{v_0 \pm \sqrt{v_0^2 - 2gs + 2gs_0}}{g}$

4. (a) The P-intercept of the function is -800 dollars, which means that the company loses $800 if zero Geegaws are sold.
   (b) Acme must make and sell between 34 and 236 Geegaws to make a true profit.
   (c) The maximum profit is $1022.50 when 135 Geegaws are made and sold. We know it is a maximum because the graph of the equation is a parabola that opens downward.

5. (a) Maximum revenue is $1,000,000 when the price is $100 per Widget.
   (b) The price needs to be less than or equal to $29.29. or greater than or equal to $170.71.

6. (a) $A = x(1000 - 2x) = 1000x - 2x^2$
   (b) $0 < x < 500$ or $(0, 500)$
   (c) The maximum area of the field is 125,000 square feet when $x = 250$ feet.
   (d) x must be between 117.7 and 382.3 feet.

Section 3.5 Solutions

Back to 3.5 Exercises

1. (a) $d = 10$ (b) $M = (1, 4)$
2. (a) $d = 5\sqrt{2} = 7.07$ (b) $M = (-\frac{1}{2}, 4\frac{1}{2})$
3. (a) $d = 7.32$ (b) $(3.6, -0.75)$
4. $(1, 12)$
5. $(-5, 2)$
6. (a) $(x-2)^2 + (y-3)^2 = 25$ (b) $(x-2)^2 + (y+3)^2 = 25$ (c) $x^2 + (y+5)^2 = 36$
7. (a) The center is $(-3, 5)$ and the radius is 2.
   (b) The center is $(4, 0)$ and the radius is $2\sqrt{7} = 5.3$.

(c) The center is $(0, 0)$ and the radius is $\sqrt{7} = 2.6$.

8. (a) $(x-1)^2 + (y-3)^2 = 4$ (b) $(x+2)^2 + (y+1)^2 = 25$

9. (a) $(x+1)^2 + (y-3)^2 = 16$, the center is $(-1, 3)$ and the radius is 4
   (b) $(x-2)^2 + (y+3)^2 = 25$, the center is $(2, -3)$ and the radius is 5
   (c) $(x-3)^2 + (y+1)^2 = 28$, the center is $(3, -1)$ and the radius is $2\sqrt{7} = 5.3$
   (d) $(x+5)^2 + y^2 = 10$, the center is $(-5, 0)$ and the radius is $\sqrt{10} = 3.2$

10. $(x-3)^2 + (y-4)^2 = 20$

11. $y = \frac{3}{4}x + \frac{25}{4}$ or $y = \frac{3}{4}x + 6\frac{1}{4}$

12. $(2, 10)$, $(2, -5)$

13. $y = -\frac{2}{3}x + 4$

14. $(x-3)^2 + (y-2)^2 = 13$

Section 3.6 Solutions

Back to 3.6 Exercises

2. (a) The solution is $(1, 8)$. If we were to graph the two equation we would see a line and a parabola intersecting in only one point.
   (b) See graph below.

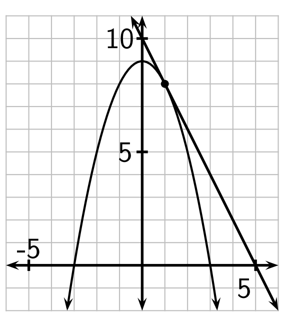
3. (a) See graph below.
   (b) The solutions are $(5, 0)$ and $(-4, 3)$.

4. (a) See graph below.
   (b) The solution is $(-2, 2)$

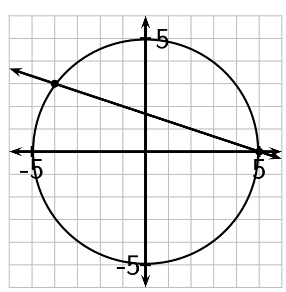
5. The solutions are $(2, -3)$ and $(4, 1)$.

7. They will break even when they produce either 34 Doo-dads or 726 Doo-dads.

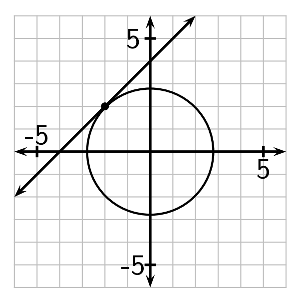
2(b)



3(a)



4(b)



Section 3.7 Solutions

Back to 3.7 Exercises

1. 151 square inches per minute

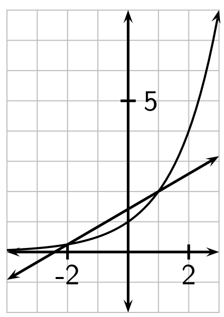
2. (a) From 4 seconds to 7 seconds the height decreased at an average rate of 32 feet per second.
   (b) The average rate of change is 0 feet per second. This is because at 2 seconds the projectile is at a height of 224 feet, on the way up, and at 7 seconds it is at 224 feet again, on the way down.

3. (a) When $x = 200$ feet, the area is 80,000 square feet, and when $x = 300$ feet, the area is 30,000 square feet.

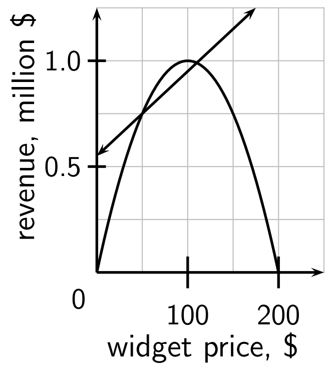
   (b) -500 square feet per foot

4. (a) $\frac{11.5}{40} = 0.29$ feet per year (b) $\frac{3}{5} = 0.6$ feet per year

5. (a) \$4000 of revenue per dollar of price (b) See graph below.



Exercise 6(a)



Exercise 5(b)

6. (a) See graph above. (b) $\frac{7}{12}$

7. 16 8. $-\frac{3}{10} = -0.9$

9. (a) $\frac{2}{3}$ (b) $\frac{2}{3}$

   (c) The answers are the same, and they are both equal to the slope of the line.

10. (a) $2x + h - 3$ (b) $6x + 3h + 5$ (c) 3

    (d) $2x + h + 7$ (e) $-2x - h + 3$ (f) $-\frac{3}{4}$

    (g) $2x + h + 4$ (h) $14x + 7h$

11. $6x + 3h$

12. (a) $(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

    (b) $f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h + 1$

    $f(x + h) - f(x) = 3x^2h + 3xh^2 + h^3 - 5h$

    $$\frac{f(x + h) - f(x)}{h} = 3x^2 + 3xh + h^2 - 5$$

    (c) $3(1)^2 + 3(1)(3) + 3^2 - 5 = 16$

13. (a) $f(x + h) - f(x) = \dfrac{-h}{x(x + h)}$ (b) $\dfrac{f(x + h) - f(x)}{h} = \dfrac{-1}{x(x + h)}$

Section 3.8 Solutions

Back to 3.8 Exercises

1. (a) $x \geq 2$ (b) $x < 11$ (c) $x < 9$

   (d) $x > 6$ (e) $x \geq -\frac{1}{5}$ (f) $x \leq \frac{4}{7}$

2. (a) $x \leq -3$ or $x \geq 7$, OR $(-\infty, -3] \cup [7, \infty)$

   (b) $x \leq -6$ or $x \geq 3$, OR $(-\infty, -6] \cup [3, \infty)$

   (c) $0 < x < 2$, OR $(0, 2)$

   (d) $1 \leq x \leq \frac{3}{2}$, OR $[1, \frac{3}{2}]$

A.4 Chapter 4 Solutions

Section 4.1 Solutions Back to 4.1 Exercises

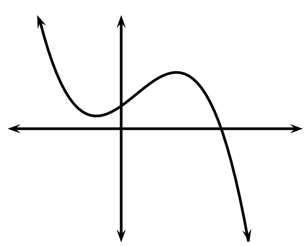
1. (a) 4 (b) $-5, 3, -7, 2$ (c) $-5x^4, 3x^2, -7x, 2$ (d) -5 (e) 2

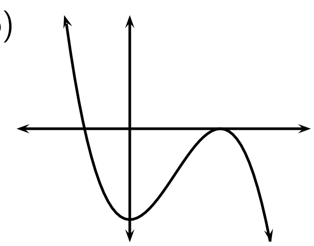
2. (a) $4, -3, -1$ (b) $3, 7, 3$ (c) $2, 1, -9$
   (d) $5, 1, 0$ (e) $1, \frac{2}{3}, -5$ (f) $0, 0, 10$

3. (a) Positive lead coefficient and constant term, even degree of four or more.
   (b) Positive lead coefficient and constant term, even degree of eight or more.
   (c) Negative lead coefficient, no constant term, even degree of four or more.
   (d) Positive lead coefficient, no constant coefficient, odd degree of three or more.

4. (a) As $x \to -\infty,\ y \to \infty$ and as $x \to \infty,\ y \to \infty$
   (b) As $x \to -\infty,\ y \to \infty$ and as $x \to \infty,\ y \to \infty$
   (c) As $x \to -\infty,\ y \to -\infty$ and as $x \to \infty,\ y \to -\infty$
   (d) As $x \to -\infty,\ y \to -\infty$ and as $x \to \infty,\ y \to \infty$

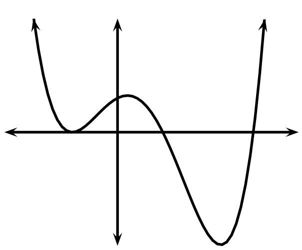
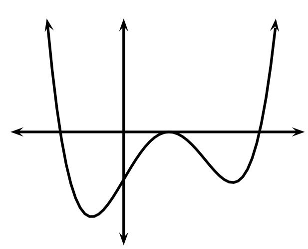
5. For those graphs that are possible, answers may vary.

   (a)  or...

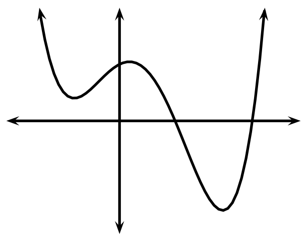
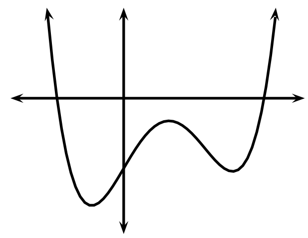
   (b)  or...

   (c) not possible

   (d) not possible

   (e)  or... 

   (f) not possible

(g)  or... 

6. Any even degree.
7. 0, 1, 2, 3, 4
8. 3, 4, 5
9. one

10. none ($y = a$, $a \neq 0$) or infinitely many ($y = 0$)

11. (a) could not, lead coefficient for graph is negative
    (b) could
    (c) could not, constant coefficient for graph must be positive
    (d) could not, degree for graph must be even and at least six

12. (a) The polynomial has neither an absolute maximum nor an absolute minimum. This is because one tail goes up "forever," and the other tail goes down "forever."
    (b) The polynomial has an absolute maximum, because both tails go up forever, so somewhere there is a lowest turning point.
    (c) Any polynomial of degree two or more can (but doesn't have to) have relative maxima or relative minima, or both. Even degree polynomials must have an absolute maximum or absolute minimum, but not both.

13. (a) $(-\infty, \infty)$ or $\mathbb{R}$ (b) $(-\infty, \infty)$ or $\mathbb{R}$ (c) $(-\infty, \infty)$ or $\mathbb{R}$
    (d) one of the forms $(-\infty, a]$ or $[a, \infty)$ for some number a

14. $y = x + 3$

15. (a) The function is increasing on $[-5, -3]$ or $(-5, -3)$.
    (b) The function is positive on $(-5, 0)$.
    (c) The function has a relative minimum of zero at $x = -5$, and a relative maximum of 21.6 at $x = -3$.

Section 4.2 Solutions Back to 4.2 Exercises

3. (b) When x is on either side of 4, $(x-4)^2$ is positive, so the graph of f "bounces off" the x-axis at $x = 4$. That is, it touches the axis but does not pass through. When x is less than 4, $(x-4)^3$ is negative, and when x is greater than 4, $(x-4)^3$ is positive. Therefore the graph of h passes through the x-axis at $x = 4$.

4. (a) The factors are $x + 3$, $x + 2$, and $x - 7$.
   (b) $x + 2$ should be squared
   (c) $P(x) = (x+3)(x+2)^2(x-7)$ The degree of this polynomial is four, and its lead coefficient is -1.

(d) The y-intercept is $P(0) = (3)(2)^2(-7) = -84$.

(f) $P(x) = \frac{1}{42}(x+3)(x+2)^2(x-7)$

6. (a) $f(x) = x^2(x-5)$ (b) $f(x) = 4(x+4)(x+2)(x-1)^2$

8. (a) $f(x) = -\frac{1}{6}(x+3)(x+2)(x-5)$ (b) $f(x) = -2x^2(x+2)(x-3)$
   (c) $f(x) = \frac{1}{10}(x+5)(x+4)(x-3)^2$ (d) $f(x) = -\frac{1}{3}(x+2)(x-3)(x-5)$

9. (a) As $x \to -\infty,\ y \to \infty$ and as $x \to \infty,\ y \to -\infty$
   (b) As $x \to -\infty,\ y \to -\infty$ and as $x \to \infty,\ y \to -\infty$
   (c) As $x \to -\infty,\ y \to \infty$ and as $x \to \infty,\ y \to \infty$
   (d) As $x \to -\infty,\ y \to \infty$ and as $x \to \infty,\ y \to -\infty$

Section 4.3 Solutions

Back to 4.3 Exercises

1. $(-\infty, -1]$ and $[0, 4]$

2. (a) $(-\infty, -3)$ and $(1, 5)$ (b) $[-3, -2]$ and $[2, 5]$
   (c) $(-5, -1)$ and $(-1, \infty)$ (d) $(-\infty, -2]$, $\{0\}$ and $[2, \infty)$
   (e) $[0, 4]$ and $[6, \infty)$ (f) $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$ OR $\{x \mid x \neq -2, 3\}$

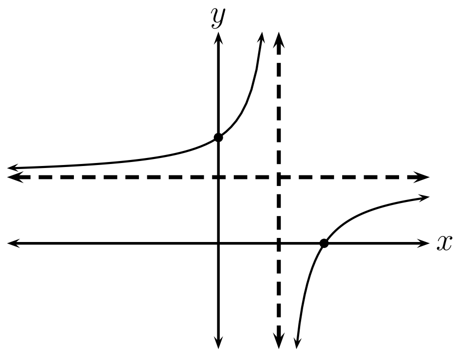
Section 4.4 Solutions

Back to 4.4 Exercises

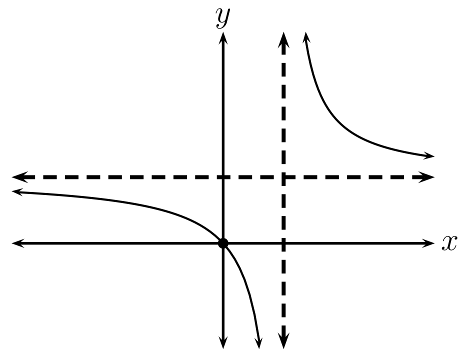
1.

(a)	(i) none	(ii) -2	(iii) $x = 3$	(iv) $y = 0$
(b)	(i) 0	(ii) 0	(iii) $x = -1$	(iv) $y = 2$
(c)	(i) none	(ii) $-\frac{3}{5}$	(iii) $x = -5$	(iv) $y = 0$
(d)	(i) 3	(ii) 6	(iii) $x = -1$	(iv) $y = -2$
(e)	(i) $-1, 3$	(ii) $\frac{3}{8}$	(iii) $x = -4,\ x = 2$	(iv) $y = 1$
(f)	(i) $-3, 5$	(ii) 15	(iii) $x = -1,\ x = 2$	(iv) $y = 2$
(g)	(i) $-4, 1$	(ii) 1	(iii) $x = -2,\ x = 2$	(iv) $y = 1$
(h)	(i) -2	(ii) $-\frac{2}{3}$	(iii) $x = -1,\ x = 3$	(iv) $y = 0$
(i)	(i) none	(ii) $\frac{4}{9}$	(iii) $x = 3$	(iv) $y = 0$
(j)	(i) $-\frac{5}{2}, -1$	(ii) $-\frac{5}{4}$	(iii) $x = -4,\ x = 1$	(iv) $y = 2$
(k)	(i) $-2, 2$	(ii) -1	(iii) $x = -1,\ x = 4$	(iv) $y = -1$
(l)	(i) -1	(ii) none	(iii) $x = 0, 5$	(iv) $y = 0$

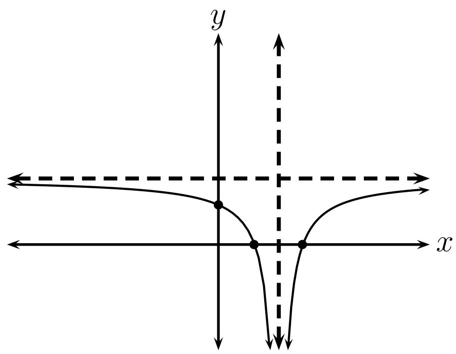
5. (a)



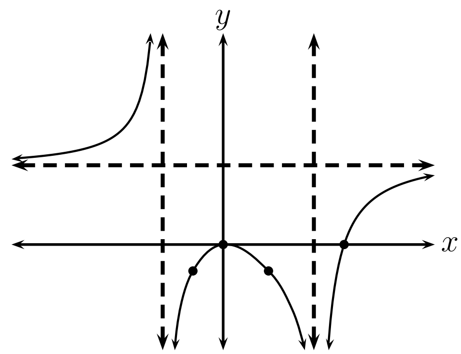
(b)



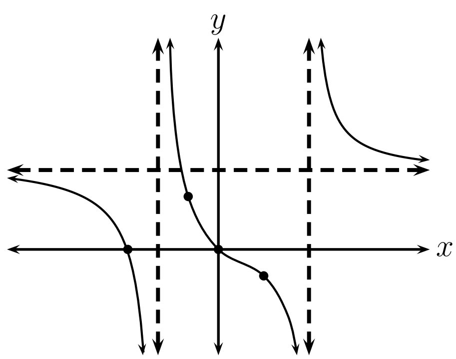
(c)



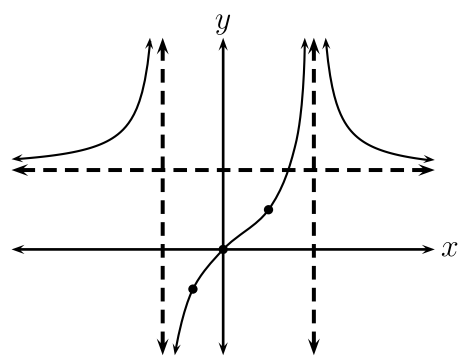
(d)



(e)



(f)



6. (a) The cost is \$40,000, the average cost per Widget is \$8.00 per Widget.

   (b) The cost is \$75,000, the average cost per Widget is \$7.50 per Widget.

   (c) When more Widgets are produced, the fixed costs of \$5000 are "spread around more."

   (d) $\bar{C} = \dfrac{7x + 5000}{x}$

   (e) The mathematical domain is all real numbers except zero. The feasible domain is $(0, \infty)$.

   (f) The vertical asymptote is $x = 0$, indicating that the number of Widgets produced each week cannot be zero.

   (g) The horizontal asymptote is $y = 7$, indicating that the average cost cannot be less than \$7.00 per Widget.

Section 4.5 Solutions Back to 4.5 Exercises

1. From top blank to bottom, $-\infty$, ∞, $-\infty$, 1

2. (a) as $x \to -3$, $y \to -\infty$, as $x \to \pm\infty$, $y \to 2$

   (b) as $x \to -2^-$, $y \to -\infty$, as $x \to -2^+$, $y \to \infty$

   as $x \to 2^-$, $y \to \infty$, as $x \to 2^+$, $y \to -\infty$

   as $x \to \pm\infty$, $y \to 0$

   (c) as $x \to 4^-$, $y \to -\infty$, as $x \to 4^+$, $y \to \infty$

   as $x \to 1$, $y \to \infty$, as $x \to \pm\infty$, $y \to 1$

3. (a) The range is $\{y \mid y \neq 0\}$ (b) Ran$(f) = (0, \infty) = \{y \mid y > 0\}$

   (c) Ran$(g) = \{y \mid y \neq 1\}$ (d) Ran$(h) = [0, \infty) = \{y \mid y \geq 0\}$

   (e) The range is $(-\infty, 0] \cup (2, \infty) = \{y \mid y \leq 0 \text{ or } y > 2\}$ (f) Ran$(g) = \mathbb{R}$

4. Equivalent set builder or interval notation answers are acceptable.

   (a) none (b) $(-\infty, 2)$ (c) $\{x \mid x \neq 2\}$

   (d) $[0, 2)$ or $(0, 2)$ (e) none (f) $(-\infty, -2) \cup (-2, 0)$

5. Equivalent set builder or interval notation answers are acceptable.

   (a) $(2, \infty)$ (b) $\{x \mid x \neq 2\}$ (c) $(0, 2)$

   (d) $\{x \mid x \neq 0, 2\}$ (e) $(-2, 0) \cup (2, \infty)$ (f) $(-\infty, -2) \cup (2, \infty)$

6. The functions for (a), (b), (c) and (e) have no minima or maxima

   (d) The function has an absolute minimum of zero at $x = 0$.

   (f) The function has a relative maximum of zero at $x = 0$.

A.5 Chapter 5 Solutions

Section 5.1 Solutions Back to 5.1 Exercises

1. (a) $(g+h)(x) = 5x+4$ (b) $(gh)(x) = 6x^2+13x-5$ (c) $\left(\dfrac{h}{g}\right)(x) = \dfrac{2x+5}{3x-1}$

   (d) $\text{Dom}(g+h) = \text{Dom}(gh) = \mathbb{R}$, $\text{Dom}\left(\dfrac{h}{g}\right) = \{x \mid x \neq \frac{1}{3}\}$

2. (a) $(g-f)(x) = -2x+2$ (b) $(fg)(x) = x^4+2x^3-4x^2-2x+3$ (c) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x+3}{x+1}$

   (d) $\text{Dom}(g-f) = \text{Dom}(fg) = \mathbb{R}$, $\text{Dom}\left(\dfrac{f}{g}\right) = \{x \mid x \neq -1, 1\}$

3. $(f-g)(x) = \dfrac{9}{x-2}$, $\text{Dom}(f-g) = \{x \mid x \neq 2\}$

4. $\left(\dfrac{f}{g}\right)(x) = \dfrac{2x+10}{x-4}$, $\text{Dom}\left(\dfrac{f}{g}\right) = \{x \mid x \neq -5, 0, 4\}$

5. (a) $(f+g)(x) = \dfrac{8x+38}{x^2-25}$, $\text{Dom}(f+g) = \{x \mid x \neq -5, 5\}$

   (b) $\left(\dfrac{g}{f}\right)(x) = \dfrac{7x+35}{x+3}$, $\text{Dom}\left(\dfrac{g}{f}\right) = \{x \mid x \neq -5, -3, 5\}$

7. Note that g and h can be switched in any of the following.

   (a) $g(x) = \dfrac{5}{x+3}$, $h(x) = \dfrac{-1}{x+2}$ (b) $g(x) = \dfrac{-2}{x-5}$, $h(x) = \dfrac{2}{x+2}$

   (c) $g(x) = \dfrac{-4}{x+1}$, $h(x) = \dfrac{3}{x-2}$ (d) $g(x) = \dfrac{7}{x+1}$, $h(x) = \dfrac{-3}{x-1}$

Section 5.2 Solutions Back to 5.2 Exercises

1. (a) $g[h(-4)] = -10$, $(g \circ h)(x) = 6x+14$ (b) $h[g(7)] = 45$, $(h \circ g)(x) = 6x+3$

2. $(f \circ g)(x) = x^2-7x+6$, $(g \circ f)(x) = x^2-5x-1$

3. $(h \circ g \circ f)(x) = 3|x-2|+1$, $(f \circ g \circ h)(x) = |3x+1|-2$

4. (a) $g(x) = x+7$, $y = (f \circ g)(x)$ (b) $h(x) = 3x-2$, $y = (h \circ f)(x)$

   (c) $h(x) = x-2$, $g(x) = \frac{1}{3}x+3$

5. $h(x) = x-2$, $g(x) = x^3$, $f(x) = \frac{1}{2}x+1$

6. (a) $f[g(5)] = 5$, $g[f(-2)] = -2$ (b) $(f \circ g)(x) = x$, $(g \circ f)(x) = x$

   (c) When we "do" g to a number, then "do" f to the result, we end up back where we started. The same thing happens if we do f first, then g.

7. (a) $h = 5t$ (b) $d = \sqrt{h^2 + 6400}$ (c) $d = \sqrt{(5t)^2 + 6400} = \sqrt{25t^2 + 6400}$

Section 5.3 Solutions

Back to 5.3 Exercises

1. (a) inverses (b) not inverses (c) inverses (d) inverses

2. (a) $g^{-1}(x) = \dfrac{5x+1}{2}$ (b) $f^{-1}(x) = \dfrac{x+2}{5}$ (c) $h^{-1}(x) = (\frac{x}{2})^3 + 1$ or $h^{-1}(x) = \frac{1}{8}x^3 + 1$

   (d) $f^{-1}(x) = \dfrac{2x}{x-4}$ (e) $h^{-1}(x) = \sqrt[3]{x-4}$ (f) $g^{-1}(x) = \dfrac{4x+3}{2x-1}$

3. (a) $f^{-1}(x) = \dfrac{x^3+1}{2}$ (b) $g^{-1}(x) = \dfrac{(x+1)^3}{2}$ (c) $h^{-1}(x) = \left(\dfrac{x+1}{2}\right)^3$

4. (a) 77°F (b) $C = \frac{5}{9}F - \frac{160}{9}$ (d) 25°C

   (e) $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}F - \frac{160}{9}$ are inverse functions.

5. (a) $f(x) = x^5$, $g(x) = \dfrac{x}{7}$, $h(x) = x + 2$ (b) $f(x) = \sqrt[3]{x}$, $g(x) = x - 1$, $h(x) = 2x$

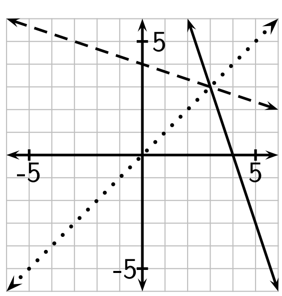
Section 5.4 Solutions

Back to 5.4 Exercises

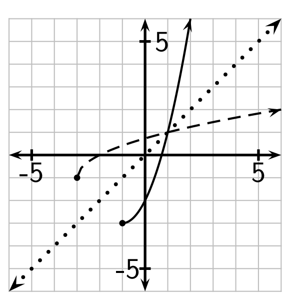
1. (b)

3. The functions graphed in (a), (d), (e) and (f) are invertible. The original graphs are shown below as dashed curves, and their inverses are given as solid curves. (Any of the "curves" might be straight lines.)

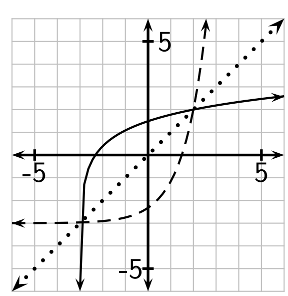
(a)

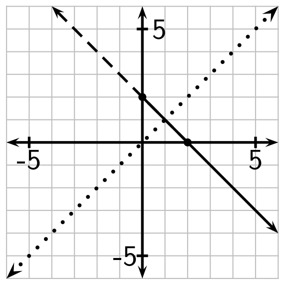


(d)



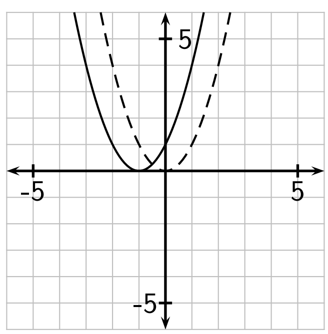
(e)

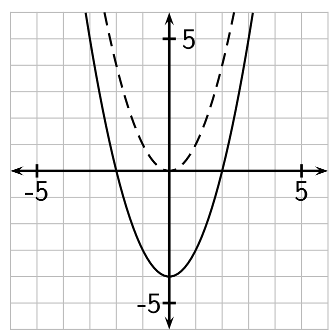


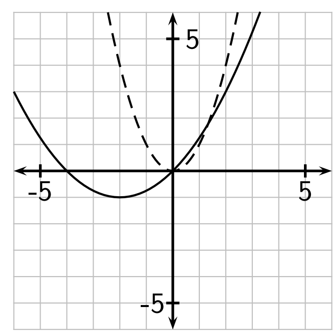
(f) 

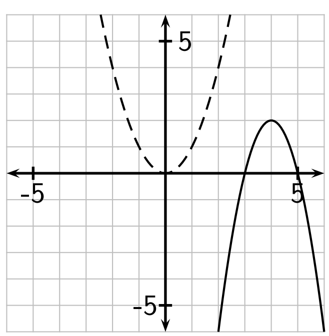
The functions graphed in (b) and (c) are not invertible. For (b) we can see that, for example, $f(-1) = f(5) \approx 2.1$, or $f(0) = f(4) \approx 4.1$. For (c), $f(0) = f(4) = 0$, $f(1) = f(3) = -1$.

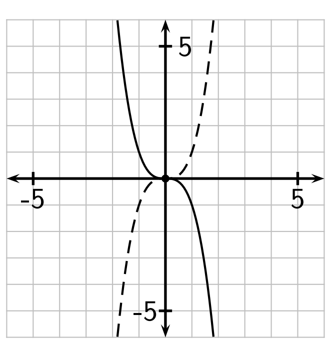
Section 5.5 Solutions Back to 5.5 Exercises

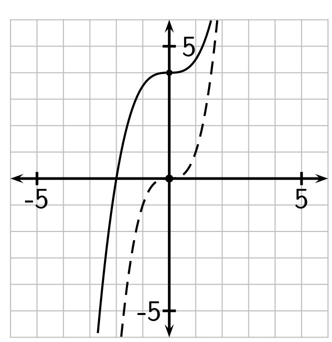
1. (a) 

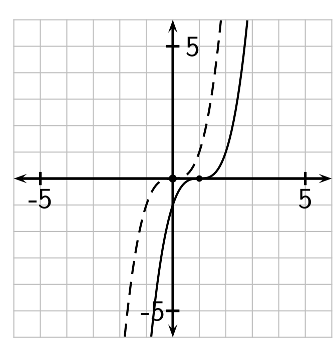
(b) 

(c) 

(d) 

2. (a) 

(b) 

(c) 

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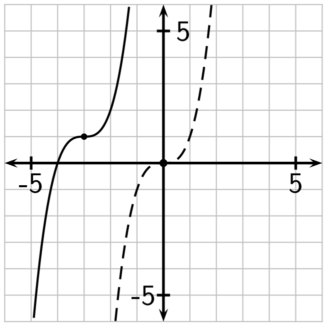

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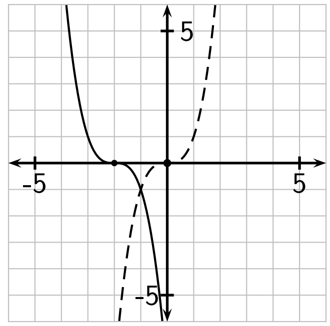
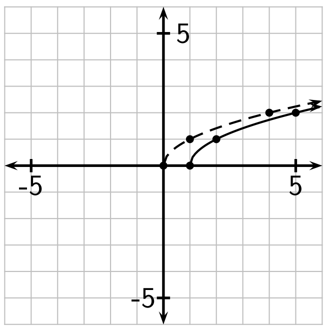
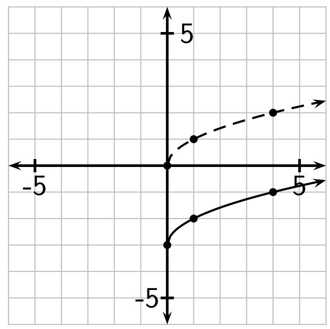


3. (a)


(b)


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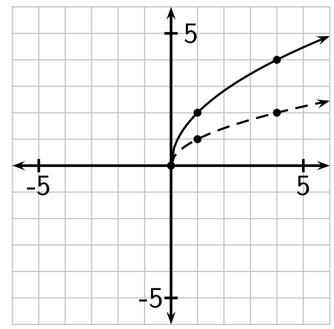

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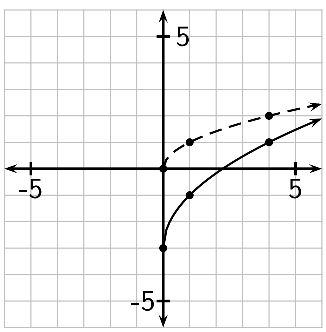

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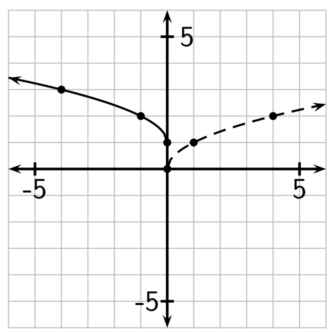

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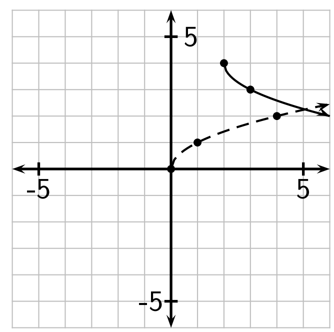
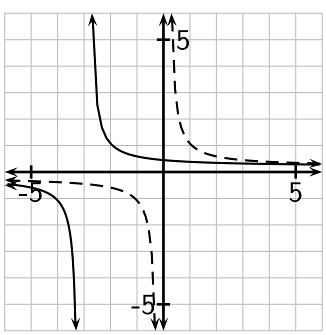


4. (a)


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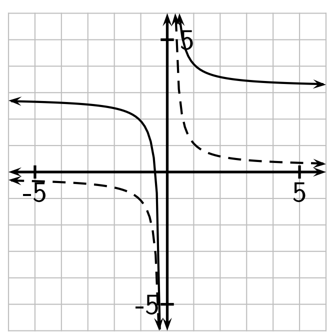

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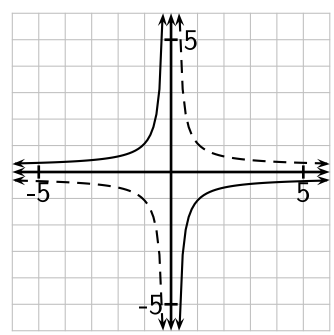

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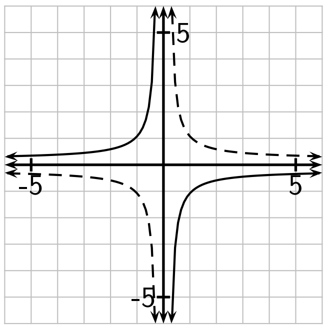

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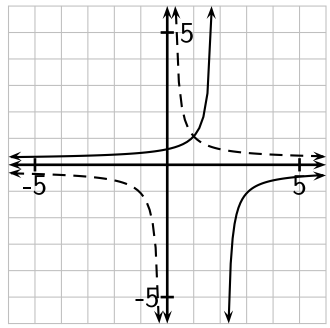

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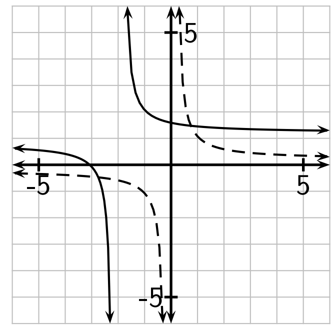
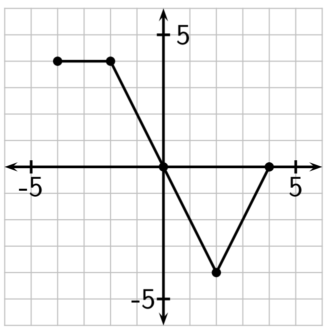


5. (a)



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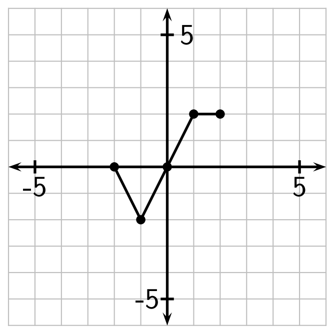

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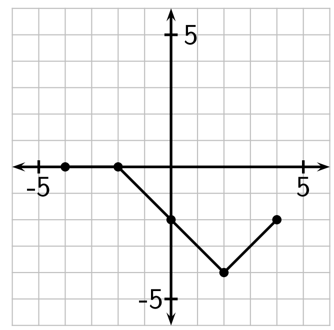

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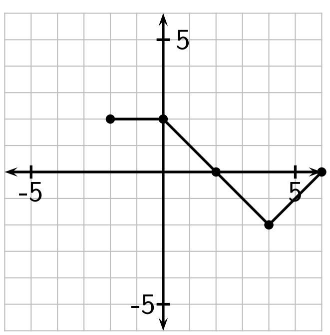

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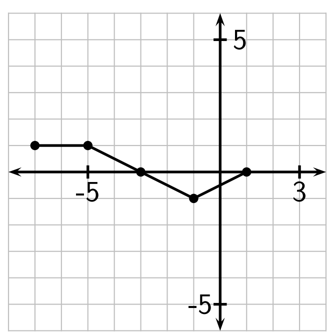

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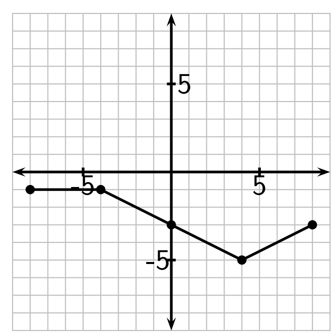

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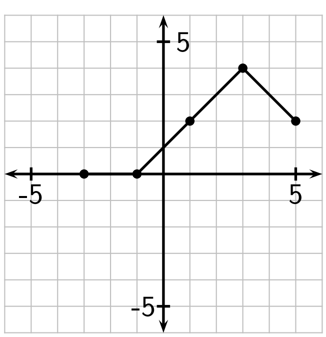

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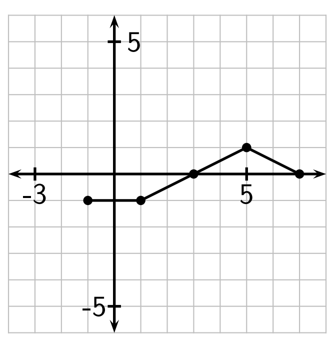
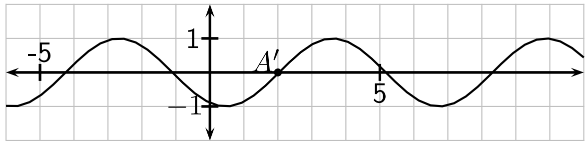


6. (a)



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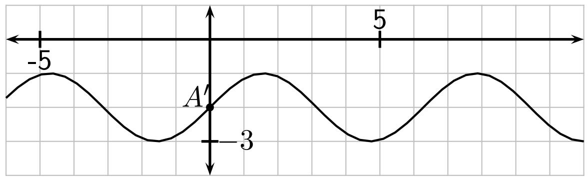

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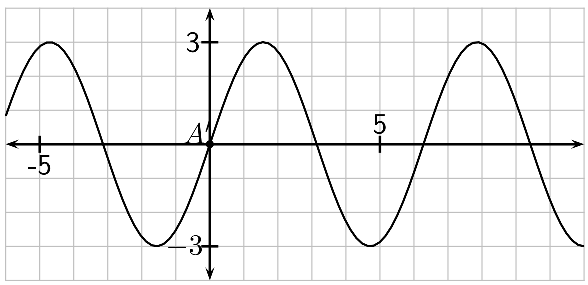
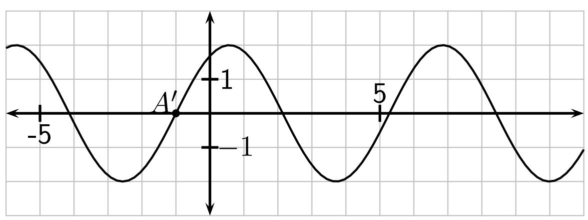


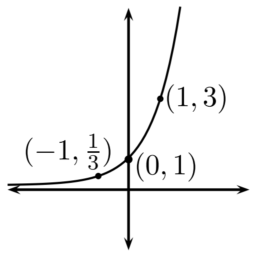
(d)

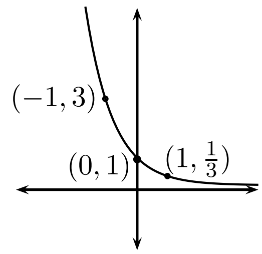


A.6 Chapter 6 Solutions

Section 6.1 Solutions Back to 6.1 Exercises

1. (a) $f(8) = 28.42$ (b) $t = 5.3$ (c) -11.76

2. (a)


(b)


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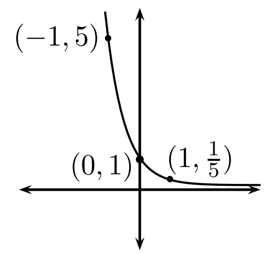
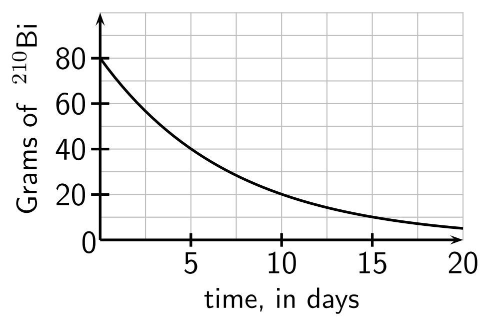


5. $x = 2.91$, $g(x) = 99.41$

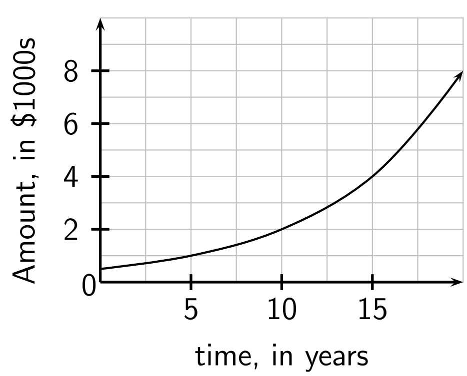
6. (a) After two days there will be four pounds left.
   (b) There will be 2 pounds left after about 3.5 days.
   (c) The average rate of change from 0 days to 5 days is -1.8 pounds per day.
   (d) The average rate of change from 2 days to 5 days is -1 pounds per day.

7. (a) See graph to the right.
   (b) There will be 30 grams of ^{210}Bi left after about seven days.



8. You began with 96 grams. (After 20 days there were 6 grams, after 15 days there were 12 grams, ...)

9. (a) See graph to the right.
   (b) You will have $3000 after about thirteen years.



Section 6.2 Solutions Back to 6.2 Exercises

1. (a) \$563.64 (b) \$2728.58 (c) \$63,363.29 (d) \$1778.78 per year

2. \$861.11

3. \$3442.82

4. (a) \$4812.00 (b) \$201.33 per year

5. \$5766.26

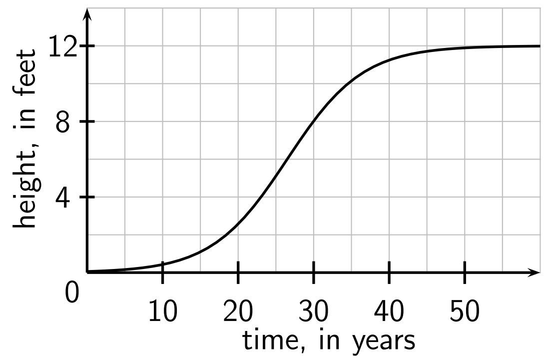
6. 6.9%

7. (a) 1321 million people (b) 22.36 million people per year

8. (a) 80.2 feet

   (b) See graph to the right.

   (c) The tree seems to be growing most rapidly when it is about 25 to 30 years old.



9. (a) 2.65 feet per year (b) 25.7 feet per year

10. $x = 3, \quad x = -3$

11. $x = 0, \quad x = 9$

Section 6.3 Solutions Back to 6.3 Exercises

1. (a) 0 (b) -4 (c) does not exist (d) $\frac{1}{2}$

2. (a) $2^{-3} = \frac{1}{8}$ (b) $10^4 = 10,000$ (c) $9^{1/2} = 3$

3. (a) $\log_4 64 = 3$ (b) $\log_{10} \frac{1}{100} = -2$ (c) $\log_8 4 = \frac{2}{3}$

4. any two of $(1,0)$, $(5,1)$, $(25,2)$, $(\frac{1}{5},-1)$, $(\frac{1}{25},-2)$, ...

5. (a) $4^{3/2} = x \quad \Longrightarrow \quad x = (\sqrt{4})^3 = 8$ (b) $9^{-2} = x \quad \Longrightarrow \quad x = \frac{1}{9^2} = \frac{1}{81}$

6. $\log_7 100p = x$

7. $4^x = \frac{1}{16} \quad \Longrightarrow \quad x = -2$

8. (a) 1.08 (b) 2.86 (c) -3.35 (d) 0.66

9. (a) 6.2 (b) 0.89 (c) 4 (d) 7.1

10. (a) $7x$ (b) x^2 (c) 12.3 (d) $x - 5$

11. (a) 1.447 (b) -0.505

12. The result is the same whether you use the common logarithm (base ten) or the natural logarithm.

13. (a) $x = \frac{2}{3}$ (b) $w = 13$ (c) $t = 7$ (d) $x = \dfrac{e^3 + 1}{2}$

(e) $x = \dfrac{5^{d-b}}{a}$ (f) $x = \dfrac{5^d - b}{a}$ (g) $b = 5^d - ax$

14. (a) $f^{-1}(x) = \dfrac{e^x - 1}{3}$ (b) $y = \log_2(x - 5)$

(c) $g^{-1}(x) = \log_3 \frac{x}{2}$ (d) $h^{-1}(x) = 2^{7x} - 1$

(e) $y = \dfrac{10^x + 4}{3}$ (f) $f^{-1}(x) = \dfrac{\log_3 x - 1}{2}$

Section 6.4 Solutions

Back to 6.4 Exercises

1. $k = 1.85$ 2. $t = 130$ 3. 14.6 years

4. (a) 8.9 years (b) 3.0 years

5. (a) 43.9 years after 1985, or in October of 2028 6. 35 years

(b) The rate of population growth has decreased since when the model was developed.

7. $I_2 = 1.3\mu\text{W/cm}^2$ 8. $I_2 = 20(10)^{-0.4d}$ 9. -3.0 dB

10. 6.46 days 11. 9.73 years 12. $k = -0.23$

13. (a) $t = 0,\ N = 500$ (b) $N_0 = 500,\ N = 500e^{kt}$ (c) $k = 0.083,\ N = 500e^{0.083t}$

(d) $N = 1354$ mice after one year

14. $A_0 = 150$, so $A = 150e^{kt}$, $k = -0.056$, so $A = 150e^{-0.056t}$, $A = 86$mg after 10 hours

Section 6.5 Solutions

Back to 6.5 Exercises

1. (a) $\log_4 x - 2\log_4 y$ (b) $3\log_5 x + \frac{1}{2}\log_5 y$

(c) $\frac{2}{3}\ln x + \frac{1}{3}\ln y$ (d) $\frac{1}{2}\ln x - 3\ln y$

2. (a) $\log x^2 y$ (b) $\log_2 \dfrac{\sqrt{x}}{y^3}$ (c) $\ln \dfrac{x}{yz}$

(d) $\log_8 \dfrac{y}{\sqrt{z^3}}$ (e) $\ln \dfrac{1}{x-1}$ (f) $6\log x$

3. (a) $x = \frac{1}{3}$ (b) $x = 3$; you should also get $x = -2$, but it "doesn't check"

(c) no solution; you should get $x = -2$ but it doesn't check

(d) $x = 4,\ x = -1$ doesn't check (e) $x = 4,\ x = -2$ doesn't check (f) $x = \frac{1}{10}$

4. $x = \frac{1}{3}$ 5. $\log_3 -5\log_3 y$ 6. $\log \sqrt{x}\, y^3$

7. $x = 1$ (-8 doesn't check) 8. $\frac{1}{2}\log x - \log y - 3\log z$

Section 6.6 Solutions Back to 6.6 Exercises

1. $t = 0.32$
2. (a) 16 g (b) 1.2 days (c) 5.8 days (d) 19.2 days
3. 7.65 hours
4. 8.8 years
5. (a) between 7 and 10.5 years (b) $N = 900e^{0.198t}$ (c) $t = 8.7$ years
6. $W = 37.5(2)^{-0.17t}$
7. $t = 0.32$
8. $t = 8.8$ years
9. $t = 13.7$ days
10. $t = 22.1$ years
11. Model: $N = 1000e^{0.14t}$, $N = 5,366$ bacteria
12. Model: $A = 25e^{-0.0012t}$, $A = 10.4$ pounds
13. $t = 18.4$ hours

A.7 Chapter 7 Solutions

Section 7.1 Solutions Back to 7.1 Exercises

1. (a) $-2y+7z=16$ (b) $7y+3z=-1$ (c) $z=2$ (d) $y=-1$ (e) $x=3$

2. (a) $(-2,1,4)$ (b) $(1,\frac{1}{2},\frac{9}{2})$ (c) $(4,3,2)$

Section 7.2 Solutions Back to 7.2 Exercises

1. (a) $\begin{bmatrix} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{bmatrix}$ $\begin{matrix} 5R_1+R_2 \longrightarrow R_2 \\ \Longrightarrow \\ -4R_1+R_3 \longrightarrow R_3 \end{matrix}$ $\begin{bmatrix} 1 & 5 & -7 & 3 \\ 0 & 28 & -36 & 15 \\ 0 & -20 & 36 & -13 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 0 & 0 \\ 0 & 6 & -5 & 2 \end{bmatrix}$ $\begin{matrix} \Longrightarrow \\ \\ 3R_2+R_3 \longrightarrow R_3 \end{matrix}$ $\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 4 & 5 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{bmatrix}$ $\begin{matrix} 2R_2+(-3)R_3 \longrightarrow R_3 \\ \text{OR} \\ (-2)R_2+3R_3 \longrightarrow R_3 \end{matrix}$ $\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 0 & 34 \text{ or } -34 & -7 \text{ or } 7 \end{bmatrix}$

2. (a) $(-4,\frac{1}{2},-4)$ (b) $(33,-4,1)$ (c) $(7,0,-2)$

3. (a) $(2,3,-1)$ (b) $(-2,1,2)$ (c) $(-1,2,1)$

Section 7.3 Solutions Back to 7.3 Exercises

1. $A^2=\begin{bmatrix} -15 & 5 \\ -3 & -14 \end{bmatrix}$ $AB=\begin{bmatrix} -10 & 0 & 25 \\ -14 & 21 & -4 \end{bmatrix}$ $AF=\begin{bmatrix} 30 & 45 \\ 0 & 12 \end{bmatrix}$

   $BC=\begin{bmatrix} -69 \\ -25 \end{bmatrix}$ $BD=\begin{bmatrix} 62 & -25 & -2 \\ -7 & 5 & -6 \end{bmatrix}$ $CE=\begin{bmatrix} -25 & 5 & -10 \\ 20 & -4 & 8 \\ -35 & 7 & -14 \end{bmatrix}$

   $DC=\begin{bmatrix} -51 \\ 27 \\ -1 \end{bmatrix}$ $D^2=\begin{bmatrix} 39 & 3 & 18 \\ -48 & 18 & -7 \\ 1 & 4 & 5 \end{bmatrix}$ $EC=\begin{bmatrix} -43 \end{bmatrix}$

   $ED=\begin{bmatrix} 37 & -2 & 13 \end{bmatrix}$ $FA=\begin{bmatrix} 3 & 9 \\ -27 & 39 \end{bmatrix}$ $FB=\begin{bmatrix} 10 & -14 & 1 \\ 6 & -42 & 63 \end{bmatrix}$

   $F^2=\begin{bmatrix} -2 & 11 \\ 66 & 75 \end{bmatrix}$

2. $23, \ -32$

3. $AI=IA=A$

4. $AB=\begin{bmatrix} -6 \\ 23 \\ -3 \end{bmatrix}$ $AX=\begin{bmatrix} 3x+0y+2z \\ -1x+4y+5z \\ 1x+1y+0z \end{bmatrix}$

5. $AC=CA=I$

Section 7.4 Solutions Back to 7.4 Exercises

1. (b) Since $AB = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, A and B are not inverses.

   (c) Since $AB = \frac{1}{14}\begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, A and B are inverses.

2. (a) $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$

3. (a) $x = -19,\ y = 13$ (b) $x = -1, y = 1$ (c) $x = 34,\ y = -13$

4. (a) $x = \frac{7}{9},\ y = -\frac{1}{9},\ z = \frac{28}{9}$ (b) $x = \frac{16}{3},\ y = \frac{16}{3},\ z = -\frac{1}{3}$ (c) $x = 2,\ y = 2,\ z = -3$

Section 7.5 Solutions Back to 7.5 Exercises

1. (a) -34 (b) 0 (c) -28 (d) 4

2. (a) $(1, 3)$ (b) $(-2, 5)$ (c) $(4, 0)$ (d) $(-1, 7)$

For Exercise 3, check your answers by substituting them into *BOTH* of the original equations. In Exercise 4 the problem is that the determinant in the bottoms of the fractions are zero.